

# GLOBAL EXISTENCE FOR THE 2D INCOMPRESSIBLE ISOTROPIC ELASTODYNAMICS FOR SMALL INITIAL DATA

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## Abstract

We establish the global existence and the asymptotic behavior for the 2D incompressible isotropic elastodynamics for sufficiently small, smooth initial data in the Eulerian coordinates formulation. The main tools used to derive the main results are, on the one hand, a modified energy method to derive the energy estimate and on the other hand, a Fourier transform method with a suitable choice of  $Z$ - norm to derive the sharp  $L^\infty$ - estimate.

This paper improves the almost global existence result of Lei-Sideris-Zhou [25] in the Eulerian coordinates formulation. We mention that the global existence of the same system but in the *Lagrangian* coordinates formulation was recently obtained by Lei in [26]. Our goal is to improve the understanding of the behavior of solutions in different coordinates using a different approach and from the point of view in frequency space.

## CONTENTS

1. Introduction	1
2. Preliminary	7
3. Energy Estimate	9
4. Linear Decay Estimate	32
5. Proof of Proposition 2.4	33
6. Asymptotic behavior of the solution	40
Appendix: Derivation of system (1.9) from system (1.7)	40
References	41

## 1. INTRODUCTION

In this paper, we consider the questions of the global existence and the asymptotic behavior for the motion of elastic waves for isotropic incompressible materials in 2D. The motion of an elastic body is described as a time dependent family of orientation preserving diffeomorphism  $x(t, \cdot)$ ,  $0 \leq t < T$ . Material point  $X$  in the reference configuration is deformed to the spatial position  $x(t, X)$  at time  $t$ . Initially, we have  $x(0, X) = X$ . We use  $X(t, x)$  to denote the inverse map of  $x(t, X)$  and we have  $X(0, x) = x$ .

Classically, the dynamic of motion can be described as a second order partial differential equation in the Lagrangian coordinates. But in the incompressible case, the equations are more conveniently formulated as a first order system of equations in the Eulerian coordinates. We can see more directly the motion of an elastic body in Eulerian coordinates. From now on, if without special annotations, the derivatives  $(\partial_t, \nabla)$  are with respect to the Eulerian coordinates  $(t, x)$ .

For the purposes of fixing notations and seeing how the incompressible condition enters the picture, we record the following lemma, which can be found in [25][Lemma 2.1]:

**Lemma 1.1.** *Given a family of deformations  $x(t, X)$ , define the velocity, deformation gradient and the displacement gradient as follows:*

$$v(t, x) = \left. \frac{dx(t, X)}{dt} \right|_{X=X(t, x)}, \quad F(t, x) = \left. \frac{\partial x(t, X)}{\partial X} \right|_{X=X(t, x)}, \quad G(t, x) = F(t, x) - I.$$

If  $x(t, X)$  is incompressible, then  $\det F(t, x) \equiv 1$ ,

$$\nabla \cdot v = 0, \quad (\nabla \cdot F^\top)_j = (\nabla \cdot G^\top)_j = \partial_i G_{i,j} = 0, \quad (1.1)$$

and

$$\partial_j G_{i,k} - \partial_k G_{i,j} = G_{m,k} \partial_m G_{i,j} - G_{m,j} \partial_m G_{i,k}, \quad i, j, k \in \{1, 2, \dots, n\}. \quad (1.2)$$

*Proof.* A detailed proof of this lemma can be found in [24][Section 2]. We only give brief remarks here. From the incompressible condition, we have the volume preserving condition, which is  $\det(F(t, x)) \equiv 1$ . After taking a derivative with respect to “ $t$ ” for this equality, we have

$$0 = \frac{d}{dt} \det(F(t, x)) = \det(F)(t, x) \text{trac}(F_t F^{-1}) \implies \text{trac}(F_t F^{-1}) = 0,$$

note that

$$x_t(t, X) = v(t, x(t, X)) \implies F_t = \nabla v F \implies 0 = \text{trac}(F_t F^{-1}) = \text{trac}(\nabla v) = \nabla \cdot v = 0.$$

It is easy to see that  $F(0, x) = I$ , hence  $\nabla \cdot F^\top(0, x) = 0$  and it can be shown that  $\nabla \cdot F^\top$  satisfies a transport equation. Since it starts from zero, then it will remain at zero for all the time.

Lastly, equality (1.2) comes from the facts that  $F_{m,j} \partial_m F_{i,k} = F_{m,k} \partial_m F_{i,j}$ , which is derived from the commutativity of material derivatives  $\partial_{X_k} \partial_{X_j} = \partial_{X_j} \partial_{X_k}$  in spatial coordinates formulation and the definition of  $G(t, x)$ .  $\square$

With the notation in Lemma 1.1, the system of incompressible isotropic elastodynamics can be formulated as follows,

$$\begin{cases} \partial_t v + v \cdot \nabla v = -\nabla p + \nabla \cdot T(F), \\ \partial_t G - \nabla v = -v \cdot \nabla G + \nabla v G, \\ \nabla \cdot v = 0, \quad \nabla \cdot G^\top = 0, \end{cases} \quad (1.3)$$

where  $T(F)$  is the Cauchy stress tensor and it is derived from the energy functional  $W(F)$  as follows,

$$T(F) := (\det F)^{-1} S(F) F^\top = (\det F)^{-1} \frac{\partial W(F)}{\partial F} F^\top, \quad (1.4)$$

where  $S(F)$  is the so-called the Piola-Kirchhoff stress. Readers can refer to [30][section 2] for the formal derivation of the system (1.3).

For a general isotropic elastodynamics, the energy functional  $W(F)$  satisfies the following relation,

$$W(F) = W(FQ) = W(QF), \quad (1.5)$$

for all rotational matrices  $Q$  such that  $Q = Q^\top$  and  $\det Q = 1$ . The first equality in (1.5) is resulted from the frame indifference and the second equality in (1.5) is resulted from the isotropy of materials.

Equality (1.5) implies that the energy functional  $W(F)$  depends on  $F$  through the principal invariants of  $FF^\top$ , which are  $\text{tr} FF^\top$  and  $\det FF^\top$  in 2D. If we denote  $\tau = \frac{1}{2} \text{tr} FF^\top$  and  $\delta = \det F$ , then

$W(F) = \widetilde{W}(\tau, \delta)$  for some smooth function  $\widetilde{W} : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ . It follows that the Piola-Kirchhoff stress has the following form,

$$S(F) = \frac{\partial W(F)}{\partial F} = \widetilde{W}_\tau(\tau, \delta)F + \widetilde{W}_\delta(\tau, \delta)\delta F^{-\top}. \quad (1.6)$$

In this paper, we mainly study the case of Hookean elasticity. We derive the corresponding system of equations first and then we will remark on the main differences between the Hookean case and general cases at the end of subsection 1.1.

The Hookean strain energy functional has the form  $W(F) = \frac{1}{2}|F|^2$ , which infers that  $\widetilde{W}(\tau, \delta) = \tau$ . Recall that  $\det F(t, x) \equiv 1$ . Hence, from (1.4) and (1.6), we know that  $T(F) = FF^\top$ . Combining this fact with the system of equations (1.3), we can derive the system of evolution for Hookean elasticity in terms of  $v$  and  $G$  as follows,

$$\begin{cases} \partial_t v - \nabla \cdot G = -\nabla p - v \cdot \nabla v + \nabla \cdot (GG^\top), \\ \partial_t G - \nabla v = -v \cdot \nabla G + \nabla v G, \\ \nabla \cdot v = 0, \quad \nabla \cdot G^\top = 0. \end{cases} \quad (1.7)$$

**1.1. Diagonalizing the system (1.7).** If one wants to analyze the system on the Fourier side, it is usually more convenient to symmetrize the system (1.7). We mainly conduct this process in this subsection.

As  $v$  and  $G^\top$  are divergence free, and we are in 2D setting, we can further reduce the system (1.7), which has six variables into a system that has three variables. Assume that  $\psi$  is the velocity potential of  $v^\perp$  and  $G_1, G_2$  are the velocity potentials of  $G_{\cdot,1}^\perp$  and  $G_{\cdot,2}^\perp$  respectively. More precisely,

$$v = (-\partial_2 \psi, \partial_1 \psi), \quad G_{\cdot,1} = (-\partial_2 G_1, \partial_1 G_1), \quad G_{\cdot,2} = (-\partial_2 G_2, \partial_1 G_2). \quad (1.8)$$

After tedious computations (readers can refer to the appendix for details), we can reduce (1.7) into the following system of equations,

$$\begin{cases} \partial_t \psi - \partial_1 G_1 - \partial_2 G_2 = \widetilde{\mathcal{N}}_0, \\ \partial_t G_1 - \partial_1 \psi = \widetilde{\mathcal{N}}_1, \\ \partial_t G_2 - \partial_2 \psi = \widetilde{\mathcal{N}}_2, \end{cases} \quad (1.9)$$

where

$$\begin{aligned} \widetilde{\mathcal{N}}_0 &= -|\nabla|^{-1}(R_1 f_1 + R_2 f_2), \quad \widetilde{\mathcal{N}}_1 = Q_{1,2}(G_1, \psi), \quad \widetilde{\mathcal{N}}_2 = Q_{1,2}(G_2, \psi), \\ f_i &= Q_{1,2}(\partial_i \psi, \psi) - Q_{1,2}(\partial_i G_1, G_1) - Q_{1,2}(\partial_i G_2, G_2), \quad i \in \{1, 2\}, \end{aligned} \quad (1.10)$$

and  $R_i = \partial_i/|\nabla|$  denotes the Riesz transform. The operator  $Q_{1,2}(\cdot, \cdot)$  in above equations is one of the celebrated null form bilinear operators that is defined as follows,

$$Q_{1,2}(f, g) := \partial_1 f \partial_2 g - \partial_2 f \partial_1 g. \quad (1.11)$$

In terms of potentials, we can reduce the constraint (1.2) as follows,

$$\partial_1 G_2 - \partial_2 G_1 = Q_{1,2}(G_2, G_1). \quad (1.12)$$

*Remark 1.2.* Note that the pressure term does not come into play in the first equation of the system (1.9). Since this equation is derived by applying *curl* on (1.7), as a result,  $\nabla p$  disappears. If one wants to know the pressure term, we can recover it from the solution  $(\psi, G_1, G_2)$  of (1.9). Here is how it works: Firstly, we derive the original velocity field  $v$  and displacement gradient  $G$  from potentials  $\psi, G_1$ , and  $G_2$ ; Secondly, we derive the following Poisson's equation by applying *div* operator on the first equation of system(1.7),

$$\Delta p = \operatorname{div}(\nabla \cdot (GG^\top) - v \cdot \nabla v); \quad (1.13)$$

Lastly, the pressure term can be derived by solving above Poisson's equation (1.13).

After diagonalizing the system (1.9), we get the following system of equations,

$$\begin{cases} \partial_t \phi_0 = \mathcal{N}_0 = R_1 \tilde{\mathcal{N}}_2 - R_2 \tilde{\mathcal{N}}_1, \\ \partial_t \Phi + i|\nabla|\Phi = \mathcal{N}_1 = \tilde{\mathcal{N}}_0 + i(R_1 \tilde{\mathcal{N}}_1 + R_2 \tilde{\mathcal{N}}_2), \end{cases} \quad (1.14)$$

where  $\tilde{\mathcal{N}}_0, \tilde{\mathcal{N}}_1$  and  $\tilde{\mathcal{N}}_2$  are defined in (1.10) and  $\phi_0$  and  $\Phi$  are defined as follows,

$$\phi_0 := |\nabla|^{-1}(\partial_1 G_2 - \partial_2 G_1) = R_1 G_2 - R_2 G_1, \quad \Phi := \psi + i(R_1 G_1 + R_2 G_2). \quad (1.15)$$

It is easy to see that we can recover  $(\psi, G_1, G_2)$  from  $(\phi_0, \Phi)$  by following identities,

$$\psi = \Re(\Phi) = \frac{\Phi + \bar{\Phi}}{2}, \quad G_1 = R_2 \phi_0 - R_1 \left( \frac{\Phi - \bar{\Phi}}{2i} \right), \quad G_2 = -R_1 \phi_0 - R_2 \left( \frac{\Phi - \bar{\Phi}}{2i} \right). \quad (1.16)$$

Therefore it is sufficient to study the system (1.14) instead.

We can rewrite the constraint equation (1.12) in terms of  $\phi_0$  and  $\Phi$  as follows,

$$\phi_0 = \mathcal{N}_2, \quad (1.17)$$

$$\mathcal{N}_2 = |\nabla|^{-1} \left[ Q_{1,2}(R_2 \phi_0, R_1 \phi_0) + \sum_{\substack{\mu \in \{+, -\} \\ i=1,2}} \frac{c_\mu}{2} Q_{1,2}(R_i \phi_0, R_i \Phi_\mu) + \sum_{\mu, \nu \in \{+, -\}} \frac{c_\mu c_\nu}{4} Q_{1,2}(R_2 \Phi_\mu, R_1 \Phi_\nu) \right],$$

where  $c_+ := -i, c_- := i, \Phi_+ := \Phi$  and  $\Phi_- := \bar{\Phi}$ . In later context, we also use  $P_+ \Phi$  to denote  $\Phi$  and use  $P_- \Phi$  to denote  $\bar{\Phi}$ .

*Remark 1.3.* Since  $\phi_0 = \mathcal{N}_2$  and  $\mathcal{N}_2$  is quadratic, intuitively speaking, the  $L^2$  norm of  $\phi_0$  decays at a rate  $1/t^{1/2}$  and the  $L^\infty$  norm of  $\phi_0$  decays at a rate  $1/t$ . We will show rigorously this point via a bootstrap argument later, but it is good to always keep this picture in mind.

*Remark 1.4.* For general cases, the system (1.3) only differs from the system (1.7), the Hookean case, at ‘‘cubic and higher’’ level in the sense of decay rate. To know this fact, we recommend interested readers to [25][section 10] for more details.

The approach used here in the Hookean case is robust enough to be applied to general isotropic incompressible elastodynamics cases. Firstly, we consider the energy estimate part. On one hand, intuitively speaking, those cubic and higher order terms will not cause any *additional* obstructions because of the higher decay rate than quadratic terms. On the other hand, to be rigorous, one usually confronts the difficulty of ‘‘losing derivatives’’ in the energy estimate because of the quasilinear nature. However, the system (1.3) indeed has the requisite symmetry properties to avoid losing derivatives, one can see this point in [30, 31]. Lastly, we consider the  $L^\infty$ -estimate part. We can estimate those additional cubic and higher order terms by the same method used to handle the cubic terms inside the equation satisfied by  $\Phi$  (see (5.1)).

**1.2. Statement of the main result.** Before introducing our main theorem, we define function spaces as follows,

$$\begin{aligned} X_k(\mathbb{R}^2) &:= \{f : \|f\|_{X_k} = \|f\|_{H^k} + \|Sf\|_{H^{\lfloor k/2 \rfloor}} + \|\Omega f\|_{H^{\lfloor k/2 \rfloor}} < \infty\}, \quad k \in \mathbb{N}, \\ Z &:= \left\{ f : \|f\|_Z = \|(1 + |\xi|)^{N_1+6} \hat{f}(\xi)\|_{L_\xi^\infty} < +\infty \right\}, \end{aligned}$$

$$Z' := \left\{ f : \|f\|_{Z'} = \|f\|_{W^{N_1+4}} < \infty \right\}, \quad Z'_1 := \left\{ f : \|f\|_{Z'_1} = \|f\|_{W^{N_1+2}} < \infty \right\},$$

$$\|f\|_{W^\gamma} := \sum_{k \in \mathbb{Z}} 2^{\gamma \max\{k, 0\}} \|P_k f\|_{L^\infty},$$

where “ $S$ ” is the scaling vector field and it is defined as  $S = t\partial_t + x_1\partial_1 + x_2\partial_2 = t\partial_t + r\partial_r$ ; and “ $\Omega$ ” is the rotational vector field and it is defined as  $\Omega = x_2\partial_1 - x_1\partial_2 = x^\perp \cdot \nabla$ .

The main result of this paper is as follows,

**Main Theorem.** *Let  $N_0 = 300, N_1 = N_0/2$ , and a fixed constant  $p_0 \in (0, 1/1000]$ , which is sufficiently small. If initial data  $(\tilde{\phi}_0, \tilde{\Phi}_0)$  satisfies the following estimate,*

$$\|(\tilde{\phi}_0, \tilde{\Phi}_0)\|_{X_{N_0}} + \|(\tilde{\phi}_0, \tilde{\Phi}_0)\|_Z = \epsilon_0 \leq \bar{\epsilon}, \quad (1.18)$$

where  $\bar{\epsilon}$  is a sufficiently small constant, and moreover, the initial data satisfies the constraint (1.17), then there exists a unique global solution  $(\phi_0, \Phi) \in C([0, \infty) : X_{N_0}(\mathbb{R}^2))$  of the initial value problem:

$$\begin{cases} \partial_t \phi_0 = \mathcal{N}_0, \\ \partial_t \Phi + i|\nabla|\Phi = \mathcal{N}_1, \\ \phi_0 = \mathcal{N}_2, \quad \phi_0(0) = \tilde{\phi}_0, \Phi(0) = \tilde{\Phi}_0. \end{cases} \quad (1.19)$$

Furthermore, we have the following estimate,

$$\sup_{t \in [0, \infty)} (1+t)^{-p_0} \|\Phi(t)\|_{X_{N_0}} + (1+t)^{1/2} \|\Phi(t)\|_{Z'} + (1+t)^{1/2-p_0} \|\phi_0(t)\|_{X_{N_0}} + (1+t) \|\phi_0(t)\|_{Z'_1} \lesssim \epsilon_0. \quad (1.20)$$

**1.3. Previous results.** The long time behavior of isotropic elastodynamics mainly follows the paradigm of nonlinear wave equation. As one can see from system (1.19), it is of quasilinear wave type equation (technically speaking, it is of half wave type).

There is an extensive literature devoted to the study of wave equations. For the purpose of giving concise introduction, we only list some representative works here. Even for a semilinear wave equation with small smooth localized initial data, John [18] showed that it can blow up in finite time. Meanwhile, if there exists “null structure” inside the nonlinearity, one might expect better behavior of the solutions. From the work of Klainerman [22] and the work of Christodoulou [4], we can see the role of “null structure”. Also, the vector field method introduced by S. Klainerman in [21] is a powerful tool to study the wave equations.

A natural question is that whether there exists null structure for general isotropic elastodynamics (not limited to the incompressible case). The compressible isotropic elastodynamics can be characterized by two families of waves: fast pressure waves and slower shear waves. In the incompressible case, the pressure wave does not present and the equations for shear waves possess an inherent null structure. So the answer for the incompressible case is yes, but the answer is not always yes for the compressible case.

For the 3D compressible elastodynamics, on the one hand, counterexamples to global existence were shown in [19] and [32]. In [19], John showed that the nontrivial radial solutions blow up for the dynamics of an isotropic homogeneous hyper-elastic medium with initial data that has sufficiently small compact support, if the equations satisfy a certain “genuine nonlinearity condition”. And in [32], Tahvildar-Zadeh showed the formation of singularities of relativistic dynamics of isotropic hyperelastic solids for large initial data. On the other hand, in [28], it was first noticed that there exists a null structure within the class of physically meaningful nonlinearities arising from the hyperelasticity assumption.

Now, let us focus on the incompressible case. We already know that there exists null structure inside the nonlinearity. Hence, we might expect better behavior of solution, but does it strong enough to guarantee global solution? In  $3D$ , it is true, see the works of Sideris-Thomases [30, 31].

Naturally, one might wonder what will happen in  $2D$ . In [25], Lei-Sideris-Zhou showed the almost global existence for small initial data. The methods used in [25] are mainly the vector field method and the Alinhac's trick (the ghost weighted energy method). To see how the ghost weighted energy method works, interested readers may refer to [25] for details.

Here come the questions: does global solution exist for the  $2D$  case? If it does, how to push the almost global existence to the global existence for the  $2D$  incompressible case? One might try to work harder on the vector field method to improve the previous result. But here, we are trying to provide another method and improve the understanding of this problem, by using a modified energy method and a Fourier transform method with an appropriate choice of  $Z$ -norm. Instead of from the point of view in physical space, we consider this problem in the frequency space. We hope that the argument developed here can shine some lights on the the  $2D$  compressible case.

We will use the bootstrap argument to derive the global existence. It naturally falls into two parts: energy estimate ( $L^2$ -type) part and the improved dispersion estimate ( $L^\infty$ -type). For the energy estimate part, we will use a modified energy method to construct an appropriate modified energy, which grows at most polynomially in time. For the improved dispersion estimate part, we will use a Fourier transform method with a suitable choice of  $Z$ -norm to get sharp  $1/t^{1/2}$  decay rate, which means the decay rates of the nonlinear solution and the linear solution are same.

The idea of modified energy method was ever used in [5] by P. Germain and N. Masmoudi, where they called it the iterated energy method. In [5], they used the Duhamel formula and did integration by parts in time once to convert the quadratic terms into cubic with price of "1/phase", which effectively equivalent to utilizing normal form transformation. Similar idea has also been used in the work of Hunter-Ifrim-Tataru-Wong [11], where they called it the modified energy method. Later, this method has been applied further to the study of water waves in the holomorphic coordinates formulation, see Hunter-Ifrim-Tataru [12] and Ifrim-Tataru [8] for details.

To make the Fourier transform method works properly, it is crucial to choose an appropriate  $Z$ -norm and get the improved  $Z$ -norm estimate, thereafter get the improved dispersion estimate to close the argument. Because the linear decay estimate we will prove in this paper has a similar structure as the one proved in the works of Ionescu-Pusateri [14, 15], inspired from their works, we choose the  $Z$ -norm used in [14].

To have a better picture of how this argument works, we strongly refer readers to the following references as warming up: Germain-Masmoudi [5], Germain-Masmoudi-Shatah [6, 7], Guo-Ionescu-Pausader [9, 10], Ionescu-Pausader [13] and Ionescu-Pusateri [14, 15, 16, 17].

Lastly, we mention the recent result of Lei [26], which derives the global existence of  $2D$  incompressible elastodynamics in *Lagrangian coordinates* formulation, the system there will be a second order wave type equation instead of a coupled system. An advantage of using *Lagrangian coordinates* is that, we can formulate the system in a nice way such that the bulk quadratic terms disappear. In the sense of decay rate, the nonlinearity is at *cubic and higher* level. However, in the *Eulerian coordinates* formulation, the bulk quadratic terms don't disappear and this is a major drawback of working in *Eulerian coordinates*.

To better understand the motion of incompressible elastic waves, it is good to know the behaviors of solutions both in *Eulerian coordinates* and in *Lagrangian coordinates*, for those interested readers, please refer to [26] for details.

**1.4. Outline:** In section 2, we fix notations and prove the main theorem by assuming Proposition 2.3 and Proposition 2.4 hold. In section 3, we first construct a modified energy and then use this modified energy to do energy estimate and prove Proposition 2.3. In section 4, we prove the linear decay estimate, which is one of the key lemmas to derive the improved dispersion estimate. In section 5, we prove Proposition 2.4. In section 6, we will describe the asymptotic behavior of solution in a lower regularity Sobolev space. Lastly, in the appendix, we show how to derive (1.9) in details.

**Acknowledgement.** The author would like to thank his Ph.D advisor Alexandru Ionescu for helpful discussions and suggestions on improving this manuscript and thank Benoit Pausader for helpful discussions. The author wants to express gratitude to Yu Deng for letting him aware of the reference [25] and helpful discussions. Lastly, the author thanks the anonymous referee for helpful suggestions to improve the presentation of this paper.

## 2. PRELIMINARY

**2.1. Notations.** Fix an even smooth bump function  $\tilde{\psi} : \mathbb{R} \rightarrow [0, 1]$  that supports in  $[-3/2, 3/2]$  and equals to 1 in  $[-5/4, 5/4]$ ,  $k \in \mathbb{Z}$ ,  $x \in \mathbb{R}^2$ , we define

$$\psi_k(x) := \tilde{\psi}(|x|/2^k) - \tilde{\psi}(|x|/2^{k-1}), \quad \psi_{\leq k}(x) = \sum_{l \leq k} \psi_l(x), \quad \psi_{\geq k}(x) = \sum_{l \geq k} \psi_l(x).$$

The frequency projection operator  $P_k$ ,  $P_{\leq k}$ , and  $P_{\geq k}$  are defined by the multipliers  $\psi_k(\xi)$ ,  $\psi_{\leq k}(\xi)$  and  $\psi_{\geq k}(\xi)$  respectively, i.e.,

$$\widehat{P_k f}(\xi) = \psi_k(\xi) \widehat{f}(\xi), \quad \widehat{P_{\leq k} f}(\xi) = \psi_{\leq k}(\xi) \widehat{f}(\xi), \quad \widehat{P_{\geq k} f}(\xi) = \psi_{\geq k}(\xi) \widehat{f}(\xi).$$

For any real number  $k \in \mathbb{R}$ , we use  $k_+$ ,  $k_+$  and  $k_-$  to denote  $k + \epsilon$ ,  $\max\{k, 0\}$  and  $\min\{k, 0\}$  respectively throughout the paper, where  $\epsilon$  is an arbitrary small constant. For any two numbers  $A$  and  $B$  and two absolute constants  $c, C$ ,  $c < C$ , we denote

$$A \sim B, \quad \text{if } cA \leq B \leq CA, \quad A \lesssim B, B \gtrsim A \quad \text{if } A \leq CB.$$

The Fourier transform of  $f$  is defined as follows,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^2} e^{-ix \cdot \xi} f(x) dx.$$

Besides  $\widehat{f}(\xi)$ , we also use  $\mathcal{F}(f)(\xi)$  to denote the Fourier transform of  $f$ . We use  $\mathcal{F}^{-1}(g)$  to denote the inverse Fourier transform of  $g$ .

For any two vectors  $\xi$  and  $\eta$ , we use notation  $\angle(\xi, \eta)$  to denote the angle from  $\eta$  to  $\xi$ . Hence  $\angle(\xi, \eta) = -\angle(\eta, \xi)$  and  $\angle(\xi, \eta) \in [-\pi, \pi]$ . When  $\angle(\xi, \eta)$  is very close to  $\pm\pi$ , i.e.,  $\xi$  and  $\eta$  are almost parallel but in the opposite direction, we say  $\angle(\xi, \eta)$  is small in the sense that  $\angle(\xi, -\eta)$  is small in this scenario. We say a quantity has  $k$  degrees of angle, if this quantity is of size  $\angle(\xi, \eta)^k$  or  $\angle(\xi, -\eta)^k$  when  $\angle(\xi, \eta)$  is small. We mention that  $\cos(\angle(\xi, \eta))$  is understood in the usual sense as  $\cos(\angle(\xi, \eta)) = \xi \cdot \eta / (|\xi||\eta|)$ .

We will use the convention that the symbol  $q(\cdot, \cdot)$  of a bilinear operator  $Q(\cdot, \cdot)$  is defined in the following sense throughout this paper,

$$\mathcal{F}[Q(f, g)](\xi) = \int_{\mathbb{R}^2} \widehat{f}(\xi - \eta) \widehat{g}(\eta) q(\xi - \eta, \eta) d\eta, \quad \text{where } f \text{ and } g \text{ are two well defined functions.}$$

**2.2. Bilinear estimate.** Define a class of symbol and its associated norms as follows,

$$\begin{aligned}\mathcal{S}^\infty &:= \{m : \mathbb{R}^4 \text{ or } \mathbb{R}^6 \rightarrow \mathbb{C}, m \text{ is continuous and } \|\mathcal{F}^{-1}(m)\|_{L^1} < \infty\}, \\ \|m\|_{\mathcal{S}^\infty} &:= \|\mathcal{F}^{-1}(m)\|_{L^1}, \quad \|m(\xi, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} := \|m(\xi, \eta)\psi_k(\xi)\psi_{k_1}(\xi - \eta)\psi_{k_2}(\eta)\|_{\mathcal{S}^\infty}, \\ \|m(\xi, \eta, \sigma)\|_{\mathcal{S}_{k,k_1,k_2,k_3}^\infty} &:= \|m(\xi, \eta, \sigma)\psi_k(\xi)\psi_{k_1}(\xi - \eta)\psi_{k_2}(\eta - \sigma)\psi_{k_3}(\sigma)\|_{\mathcal{S}^\infty}.\end{aligned}$$

**Lemma 2.1.** *Given  $m, m' \in \mathcal{S}^\infty$  and two well defined functions  $f_1, f_2$ , and  $f_3$ , then the following estimates hold,*

$$\begin{aligned}\|m \cdot m'\|_{\mathcal{S}^\infty} &\lesssim \|m\|_{\mathcal{S}^\infty} \|m'\|_{\mathcal{S}^\infty}, \\ \|\mathcal{F}^{-1}(\int_{\mathbb{R}^2} m(\xi, \eta) \widehat{f}_1(\xi - \eta) \widehat{f}_2(\eta) d\eta)(x)\|_{L^r} &\lesssim \|m\|_{\mathcal{S}^\infty} \|f_1\|_{L^p} \|f_2\|_{L^q}, \quad \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \\ \|\mathcal{F}^{-1}(\int_{\mathbb{R}^2 \times \mathbb{R}^2} m(\xi, \eta, \sigma) \widehat{f}_1(\xi - \eta) \widehat{f}_2(\eta - \sigma) \widehat{f}_3(\sigma) d\eta d\sigma)(x)\|_{L^s} &\lesssim \|m\|_{\mathcal{S}^\infty} \|f_1\|_{L^{p'}} \|f_2\|_{L^{q'}} \|f_3\|_{L^{r'}},\end{aligned}$$

where  $p', q', r'$ , and  $s$  satisfy  $1/s = 1/p' + 1/q' + 1/r'$ .

*Proof.* The proof is standard, or one can see [15][Lemma 5.2].  $\square$

To estimate the  $\mathcal{S}_{k,k_1,k_2}^\infty$  or the  $\mathcal{S}_{k,k_1,k_2,k_3}^\infty$  norms of symbols, we constantly use the following lemma .

**Lemma 2.2.** *For  $i \in \{2, 3\}$ , if  $f : \mathbb{R}^{2i} \rightarrow \mathbb{C}$  is a smooth function and  $k_1, \dots, k_i \in \mathbb{Z}$ , then the following estimate holds,*

$$\left\| \int_{\mathbb{R}^{2i}} f(\xi_1, \dots, \xi_i) \prod_{j=1}^i e^{ix_j \cdot \xi_j} \psi_{k_j}(\xi_j) d\xi_1 \cdots d\xi_i \right\|_{L_{x_1, \dots, x_i}^1} \lesssim \sum_{m=0}^{2i+1} \sum_{j=1}^i 2^{mk_j} \|\partial_{\xi_j}^m f\|_{L^\infty}. \quad (2.1)$$

*Proof.* Let's first consider the case when  $i = 2$ . Through scaling, it is sufficient to prove above estimate for the case  $k_1 = k_2 = 0$ . From the integration by parts in  $\xi_1$  and  $\xi_2$ , we have the following pointwise estimate

$$(1 + |x_1| + |x_2|)^5 \left| \int_{\mathbb{R}^2 \times \mathbb{R}^2} e^{ix_1 \cdot \xi_1} e^{ix_2 \cdot \xi_2} f(\xi_1, \xi_2) \psi_0(\xi_1) \psi_0(\xi_2) d\xi_1 d\xi_2 \right| \lesssim \sum_{m=0}^5 [\|\partial_{\xi_1}^m f\|_{L^\infty} + \|\partial_{\xi_2}^m f\|_{L^\infty}],$$

which is sufficient to finish the proof of (2.1). We can prove the case when  $i = 3$  very similarly, hence we omit the details here.  $\square$

**2.3. Bootstrap assumption and proof of the main theorem.** We use the bootstrap argument to prove the main theorem. The bootstrap assumption is stated as follows,

$$\begin{aligned}& \sup_{t \in [0, T]} (1+t)^{-p_0} \|\Phi(t)\|_{X_{N_0}} + (1+t)^{1/2-p_0} \|\phi_0(t)\|_{X_{N_0}} \\ & + (1+t)^{1/2} \|\Phi\|_{Z'} + (1+t) \|\phi_0\|_{Z'_1} + (1+t)^{-2p_0} \|\Phi(t)\|_Z \lesssim \epsilon_1 := \epsilon_0^{5/6}.\end{aligned} \quad (2.2)$$

Since the size of initial data is  $\epsilon_0$ , from the continuity of solution, we know the existence of  $T > 0$  in above bootstrap assumption. In later context, without further annotations, the solution is considered in the time interval  $[0, T]$  and satisfies above estimate. In the sense of decay rate, we will call  $\phi_0$  itself as a quadratic term.

In section 3, we will prove the following proposition,



**Proposition 2.3** (Energy estimate). *Under the bootstrap assumption (2.2), we have*

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|(\phi_0, \Phi)\|_{X_{N_0}} \lesssim \epsilon_0 + \epsilon_1^2 \lesssim \epsilon_0. \quad (2.3)$$

*Proof.* It follows from the result of Lemma 3.6 directly.  $\square$

In section 5, we will prove the following proposition,

**Proposition 2.4.** *Under the bootstrap assumption (2.2) and the energy estimate (2.3), we can derive the following improved estimates,*

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|\Phi(t)\|_{X_{N_0}} + (1+t)^{1/2} \|\Phi(t)\|_{Z'} + (1+t)^{-2p_0} \|\Phi\|_Z \lesssim \epsilon_0 + \epsilon_1^2 \lesssim \epsilon_0, \quad (2.4)$$

$$\sup_{t \in [0, T]} (1+t)^{1/2-p_0} \|\phi_0(t)\|_{X_{N_0}} + (1+t) \|\phi_0(t)\|_{Z'_1} \lesssim (\epsilon_0 + \epsilon_1)^2 \lesssim \epsilon_0^2. \quad (2.5)$$

*Proof of the Main Theorem.* Combining results in above two propositions, it is easy to see that, under the bootstrap assumption (2.2), we have

$$\begin{aligned} & \sup_{t \in [0, T]} (1+t)^{-p_0} \|\Phi(t)\|_{X_{N_0}} + (1+t)^{1/2-p_0} \|\phi_0(t)\|_{X_{N_0}} + (1+t)^{-2p_0} \|\Phi\|_Z \\ & + (1+t)^{1/2} \|\Phi(t)\|_{Z'} + (1+t) \|\phi_0(t)\|_{Z'_1} \lesssim \epsilon_0. \end{aligned} \quad (2.6)$$

Therefore, we can keep iterating the local result and extend the time interval of existence to the full time interval  $[0, +\infty)$ , i.e., solution exists globally. Moreover, the desired estimate (1.20) holds.  $\square$

### 3. ENERGY ESTIMATE

This section is devote to prove Proposition 2.3. Firstly, we identify null structures inside the system, which are the bases of doing this argument. Secondly, we identify the most problematic terms, which help us to figure out how to construct a modified energy. Finally, we use this modified energy to do energy estimate and finish the proof of Proposition 2.3.

**3.1. Identifying null structures inside the system.** The goal of this subsection is to check the symbols of quadratic terms very carefully to see whether there exist null structures and how “strong” null structures are.

Base on the input types inside the quadratic terms, we decompose the nonlinearities  $\mathcal{N}_0$  and  $\mathcal{N}_1$  and the constraint  $\mathcal{N}_2$  as follows,

$$\begin{aligned} \mathcal{N}_0 &= \sum_{\mu \in \{+, -\}} Q_{0, \mu}(\phi_0, \Phi_\mu) + \sum_{\mu, \nu \in \{+, -\}} Q_{\mu, \nu}(\Phi_\mu, \Phi_\nu), \\ \mathcal{N}_1 &= \tilde{Q}_{0,0}(\phi_0, \phi_0) + \sum_{\mu \in \{+, -\}} \tilde{Q}_{0, \mu}(\phi_0, \Phi_\mu) + \sum_{\mu, \nu \in \{+, -\}} \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu), \\ \mathcal{N}_2 &= \tilde{Q}_{0,0}^1(\phi_0, \phi_0) + \sum_{\mu \in \{+, -\}} \tilde{Q}_{0, \mu}^1(\phi_0, \Phi_\mu) + \sum_{\mu, \nu \in \{+, -\}} \tilde{Q}_{\mu, \nu}^1(\Phi_\mu, \Phi_\nu), \end{aligned}$$

where

$$Q_{0, \mu}(\phi_0, \Phi_\mu) = \frac{1}{2|\nabla|} \left( Q_{1,2}(|\nabla| \phi_0, \Phi_\mu) - Q_{1,2}(R_1 \phi_0, \partial_1 \Phi_\mu) - Q_{1,2}(R_2 \phi_0, \partial_2 \Phi_\mu) \right), \quad \mu \in \{+, -\},$$

$$\begin{aligned}
Q_{\mu,\nu}(\Phi_\mu, \Phi_\nu) &= \frac{c_\mu}{4|\nabla|} \left( Q_{1,2}(R_1\Phi_\mu, \partial_2\Phi_\nu) - Q_{1,2}(R_2\Phi_\mu, \partial_1\Phi_\nu) \right), \quad \mu, \nu \in \{+, -\}, \\
\tilde{Q}_{0,0}(\phi_0, \phi_0) &= \sum_{i=1,2} \frac{R_i}{|\nabla|} \left( Q_{1,2}(\partial_i R_2\phi_0, R_2\phi_0) + Q_{1,2}(\partial_i R_1\phi_0, R_1\phi_0) \right), \\
\tilde{Q}_{0,\mu}(\phi_0, \Phi_\mu) &= \sum_{i=1,2} \frac{c_\mu R_i}{2|\nabla|} \left( Q_{1,2}(\partial_i R_1\phi_0, R_2\Phi_\mu) - Q_{1,2}(\partial_i R_2\phi_0, R_1\Phi_\mu) + \right. \\
&\quad \left. Q_{1,2}(\partial_i R_2\Phi_\mu, R_1\phi_0) - Q_{1,2}(\partial_i R_1\Phi_\mu, R_2\phi_0) \right) + \frac{i}{2|\nabla|} \left( Q_{1,2}(R_2\phi_0, \partial_1\Phi_\mu) - Q_{1,2}(R_1\phi_0, \partial_2\Phi_\mu) \right), \quad (3.1)
\end{aligned}$$

$$\begin{aligned}
\tilde{Q}_{\mu,\nu}(\Phi_\mu, \Phi_\nu) &= \sum_{i,j=1,2} \frac{R_i}{4|\nabla|} \left( c_\mu c_\nu Q_{1,2}(\partial_i R_j\Phi_\mu, R_j\Phi_\nu) - Q_{1,2}(\partial_i\Phi_\mu, \Phi_\nu) \right) + \\
&\quad \frac{i c_\mu}{4|\nabla|} \left( Q_{1,2}(|\nabla|\Phi_\mu, \Phi_\nu) - Q_{1,2}(R_1\Phi_\mu, \partial_1\Phi_\nu) - Q_{1,2}(R_2\Phi_\mu, \partial_2\Phi_\nu) \right), \quad \mu, \nu \in \{+, -\}, \quad (3.2)
\end{aligned}$$

$$\tilde{Q}_{0,0}^1(\phi_0, \phi_0) = \frac{1}{2|\nabla|} [Q_{1,2}(R_2\phi_0, R_1\phi_0) - Q_{1,2}(R_1\phi_0, R_2\phi_0)],$$

$$\tilde{Q}_{0,\mu}^1(\phi_0, \Phi_\mu) = \sum_{i=1}^2 \frac{c_\mu}{2} |\nabla|^{-1} Q_{1,2}(R_i\phi_0, R_i\Phi_\mu),$$

$$\tilde{Q}_{\mu,\nu}^1(\Phi_\mu, \Phi_\nu) = \frac{c_\mu c_\nu}{8} |\nabla|^{-1} [Q_{1,2}(R_2\Phi_\mu, R_1\Phi_\nu) - Q_{1,2}(R_1\Phi_\mu, R_2\Phi_\nu)]. \quad (3.3)$$

After tedious calculations, we can show that the associated symbols of above bilinear operators are given as follows,

$$m_{0,\mu}(\xi - \eta, \eta) = -\frac{\xi \cdot (\xi - \eta)}{4\pi|\xi||\xi - \eta|} ((\xi - \eta) \times \eta), \quad m_{\mu,\nu}(\xi - \eta, \eta) = \frac{c_\mu}{8\pi|\xi|} \frac{1}{|\xi - \eta|} ((\xi - \eta) \times \eta)^2, \quad (3.4)$$

$$\tilde{m}_{0,\mu}(\xi - \eta, \eta) = -\frac{c_\mu}{4\pi|\xi|^2} \left( \frac{\xi \cdot (\xi - \eta)}{|\xi - \eta||\eta|} - \frac{\xi \cdot \eta}{|\xi - \eta||\eta|} \right) ((\xi - \eta) \times \eta)^2 - \frac{i}{4\pi|\xi||\xi - \eta|} ((\xi - \eta) \times \eta)^2, \quad (3.5)$$

$$\tilde{m}_{\mu,\nu}(\xi - \eta, \eta) = \left( -\frac{c_\mu c_\nu}{8\pi|\xi|} \frac{\xi \cdot (\xi - \eta)}{|\xi|} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta|} - \frac{\xi \cdot (\xi - \eta)}{8\pi|\xi|^2} - \frac{i c_\mu \xi \cdot (\xi - \eta)}{8\pi|\xi||\xi - \eta|} \right) ((\xi - \eta) \times \eta), \quad (3.6)$$

$$\tilde{m}_{0,0}(\xi - \eta, \eta) = -\frac{\xi \cdot (\xi - \eta)}{2\pi|\xi|^2} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta|} ((\xi - \eta) \times \eta), \quad \tilde{m}_{0,0}^1(\xi - \eta, \eta) = \frac{-((\xi - \eta) \times \eta)^2}{4\pi|\xi||\xi - \eta||\eta|}, \quad (3.7)$$

$$\tilde{m}_{0,\mu}^1(\xi - \eta, \eta) = -\frac{c_\mu}{4\pi} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta||\xi|} ((\xi - \eta) \times \eta), \quad \tilde{m}_{\mu,\nu}^1(\xi - \eta, \eta) = \frac{c_\mu c_\nu}{16\pi} \tilde{m}_{0,0}^1(\xi - \eta, \eta). \quad (3.8)$$

Recall that  $c_+ = -i$  and  $c_- = i$  as defined in the introduction.

As an example, we will show detail computations for (3.1). which is very typical. All other cases can be computed in the same way. From explicit formula in (3.1), we have

$$\begin{aligned}
\tilde{m}_{0,\mu}(\xi - \eta, \eta) &= \sum_{j=1,2} \frac{i c_\mu \xi_j}{4\pi|\xi|^2} \left[ i(\xi - \eta)_j [(\xi - \eta) \times \eta] \frac{(\xi - \eta)_1 \eta_2 - (\xi - \eta)_2 \eta_1}{|\xi - \eta||\eta|} \right. \\
&\quad \left. - i\eta_j [(\xi - \eta) \times \eta] \frac{(\xi - \eta)_1 \eta_2 - (\xi - \eta)_2 \eta_1}{|\xi - \eta||\eta|} \right] + \frac{i}{4\pi|\xi|} \left[ [(\xi - \eta) \times \eta] \frac{(\xi - \eta)_2 \eta_1 - (\xi - \eta)_1 \eta_2}{|\xi - \eta|} \right]
\end{aligned}$$

$$= \frac{-c_\mu}{4\pi|\xi|^2} \left[ \frac{\xi \cdot (\xi - \eta)}{|\xi - \eta||\eta|} - \frac{\xi \cdot \eta}{|\xi - \eta||\eta|} \right] \left( (\xi - \eta) \times \eta \right)^2 - \frac{i}{4\pi|\xi||\xi - \eta|} \left( (\xi - \eta) \times \eta \right)^2,$$

therefore (3.5) holds.

From above detailed formulas of symbols, we can see that all symbols vanishes when  $(\xi - \eta) \parallel \eta$ . Hence, indeed, there are null structures inside nonlinearities. But does those null structures, especially the one in the symbol  $\tilde{m}_{\mu,\nu}(\cdot, \cdot)$ , strong enough? The answer depends on how strong we need them to be.

In the later modified energy estimate part, we will see that it is very crucial to gain at least two degrees of angle for  $\tilde{m}_{\mu,\nu}(\cdot, \cdot)$  in certain scenarios. Otherwise, the symbol will be singular after dividing the phase.

In the following, we will show that we can gain one more degree of angle from symmetries. However, this angle depends on the fact that whether  $(\xi - \eta)$  and  $\eta$  are in the same direction. To be more precise, we divide into different cases based on the types of phases, which are determined by the types of quadratic terms. More precisely, the phases are defined as follows,

$$\Phi_{\mu,\nu}(\xi, \eta) := |\xi| - \mu|\xi - \eta| - \nu|\eta|, \quad \mu, \nu \in \{+, -\}.$$

Before we proceed, we mention that the types of phase and the discussion here are also related to the normal form transformations that we will do later (see also subsection 3.2).

Without loss of generality, we assume that  $0 < |\eta| \leq |\xi - \eta|$  in the following,

- (i) For the phase of type  $|\xi| - |\xi - \eta| - |\eta|$ , the corresponding symbol is  $\tilde{m}_{+,+}(\cdot, \cdot)$ . In this case, the phase vanishes when  $\angle(\xi, \eta) = \angle(\xi - \eta, \eta) = 0$  and  $|\xi|$  has comparable size of  $|\xi - \eta|$ .
- (ii) For the phase of type  $|\xi| - |\xi - \eta| + |\eta|$ , the corresponding symbol is  $\tilde{m}_{+,-}(\cdot, \cdot)$ . In this case, the phase vanishes when  $\angle(\xi, -\eta) = \angle(\xi - \eta, -\eta) = 0$ .
- (iii) For the phase of type  $|\xi| + |\xi - \eta| + |\eta|$  or  $|\xi| + |\xi - \eta| - |\eta|$ , it does not vanish and has a lower bound regardless whether  $\xi$  and  $\eta$  are parallel or not. For this case, it is not necessary to gain two degrees of angle.

For case (i), since two inputs of  $\tilde{Q}_{+,+}(\cdot, \cdot)$  are of the same type, we can utilize self symmetry to see that the symbol of bilinear form  $\tilde{Q}_{+,+}(\cdot, \cdot)$  is also given as follows,

$$\begin{aligned} \tilde{m}'_{+,+}(\xi - \eta, \eta) &:= [\tilde{m}_{+,+}(\xi - \eta, \eta) + \tilde{m}_{+,+}(\eta, \xi - \eta)]/2 \\ &= \left( -\frac{c_+c_+}{16\pi|\xi|} \frac{\xi \cdot (\xi - \eta)}{|\xi|} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta|} - \frac{\xi \cdot (\xi - \eta)}{16\pi|\xi|^2} - \frac{ic_+ \xi \cdot (\xi - \eta)}{16\pi|\xi||\xi - \eta|} \right) \left( (\xi - \eta) \times \eta \right) \\ &\quad + \left( -\frac{c_+c_+}{16\pi|\xi|} \frac{\xi \cdot \eta}{|\xi|} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta|} - \frac{\xi \cdot \eta}{16\pi|\xi|^2} - \frac{ic_+ \xi \cdot \eta}{16\pi|\xi||\eta|} \right) \left( \eta \times (\xi - \eta) \right) \\ &= \underbrace{\frac{-\xi \cdot (\xi - 2\eta)}{16\pi|\xi|^2} (1 - \cos(\angle(\xi - \eta, \eta))) \left( (\xi - \eta) \times \eta \right)}_{\text{three degrees of angle } \angle(\xi - \eta, \eta)} - \underbrace{\frac{\xi}{16\pi|\xi|} \cdot \left( \frac{\xi - \eta}{|\xi - \eta|} - \frac{\eta}{|\eta|} \right) \left( (\xi - \eta) \times \eta \right)}_{\text{two degrees of angle } \angle(\xi - \eta, \eta)}. \end{aligned} \tag{3.9}$$

For case (ii), we couple term  $\tilde{Q}_{+,-}(\Phi, \bar{\Phi})$  with term  $\tilde{Q}_{-,-}(\bar{\Phi}, \Phi)$  and define  $\tilde{Q}_{+,-}(\Phi, \bar{\Phi}) := \tilde{Q}_{+,-}(\Phi, \bar{\Phi}) + \tilde{Q}_{-,-}(\bar{\Phi}, \Phi)$ . Its corresponding symbol is given as follows,

$$\begin{aligned} \tilde{m}'_{+,-}(\xi - \eta, \eta) &= \tilde{m}_{+,-}(\xi - \eta, \eta) + \tilde{m}_{-,-}(\eta, \xi - \eta) \\ &= \left( -\frac{c_+c_-}{8\pi|\xi|} \frac{\xi \cdot (\xi - \eta)}{|\xi|} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta|} - \frac{\xi \cdot (\xi - \eta)}{8\pi|\xi|^2} - \frac{ic_+ \xi \cdot (\xi - \eta)}{8\pi|\xi||\xi - \eta|} \right) \left( (\xi - \eta) \times \eta \right) \end{aligned}$$

$$\begin{aligned}
& + \left( -\frac{c_- c_+}{8\pi|\xi|} \frac{\xi \cdot \eta}{|\xi|} \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta||\eta|} - \frac{\xi \cdot \eta}{8\pi|\xi|^2} - \frac{ic_- \xi \cdot \eta}{8\pi|\xi||\eta|} \right) (\eta \times (\xi - \eta)) \\
& = \frac{-\xi \cdot (\xi - 2\eta)}{8\pi|\xi|^2} \underbrace{(1 + \cos(\angle(\xi - \eta, \eta)))}_{\text{three degrees of angle } \angle(\xi - \eta, -\eta)} \left( (\xi - \eta) \times \eta \right) - \frac{\xi}{8\pi|\xi|} \cdot \underbrace{\left( \frac{\xi - \eta}{|\xi - \eta|} + \frac{\eta}{|\eta|} \right)}_{\text{two degrees of angle } \angle(\xi - \eta, -\eta)} \left( (\xi - \eta) \times \eta \right).
\end{aligned} \tag{3.10}$$

To make notations consistent, we define

$$\tilde{m}'_{-, -}(\xi - \eta, \eta) := \tilde{m}_{-, -}(\xi - \eta, \eta), \quad \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu) := \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu), \quad (\mu, \nu) \in \{(+, +), (-, -)\}.$$

To sum up, we can reformulate the equation satisfied by  $\Phi$  as follows,

$$\partial_t \Phi + i|\nabla|\Phi = \mathcal{N}_1 = \tilde{Q}_{0,0}(\phi_0, \phi_0) + \sum_{\mu \in \{+, -\}} \tilde{Q}_{0,\mu}(\phi_0, \Phi_\mu) + \sum_{(\mu, \nu) \in \mathcal{S}} \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu), \tag{3.11}$$

where  $\mathcal{S} := \{(+, +), (+, -), (-, -)\}$ . Inside the symbol of  $\tilde{Q}_{+, +}(\cdot, \cdot)$ , we can gain two degrees of angle when frequencies of two inputs are parallel and in the *same direction*. Inside the symbol of  $\tilde{Q}_{+, -}(\cdot, \cdot)$ , we can gain two degrees of angle when frequencies of two inputs are parallel and in the *opposite direction*. We can always gain one degree of angle regardless whether two frequencies in the same direction or not.

**3.2. Normal form transformation.** The first step of constructing the modified energy is to find out the normal form transformation. More precisely, we are looking for a normal form transformation  $\Phi \rightarrow \tilde{\Phi}$ , such that the equation satisfied by  $\tilde{\Phi}$  is cubic and higher. Let

$$\tilde{\Phi} = \Phi + \sum_{(\mu, \nu) \in \mathcal{S}} A_{\mu, \nu}(\Phi_\mu, \Phi_\nu), \tag{3.12}$$

where  $A_{\mu, \nu}(\cdot, \cdot)$ ,  $(\mu, \nu) \in \mathcal{S}$ , is an unknown bilinear operators to be determined. Recall the equation (3.11) satisfied by  $\Phi$ , then we have the following,

$$\begin{aligned}
\partial_t \tilde{\Phi} + i|\nabla|\tilde{\Phi} &= \sum_{(\mu, \nu) \in \mathcal{S}} \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu) + \sum_{(\mu, \nu) \in \mathcal{S}} i|\nabla|A_{\mu, \nu}(\Phi_\mu, \Phi_\nu) - \\
&\quad \left( \sum_{(\mu, \nu) \in \mathcal{S}} A_{\mu, \nu}(ia_\mu |\nabla|\Phi_\mu, \Phi_\nu) + A_{\mu, \nu}(\Phi_\mu, ia_\nu |\nabla|\Phi_\nu) \right) + \text{cubic and higher},
\end{aligned}$$

where  $a_+ = 1$  and  $a_- = -1$ . To cancel out quadratic terms, it is sufficient if the following equality holds for all  $(\mu, \nu) \in \mathcal{S}$ ,

$$\tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu) + i|\nabla|A_{\mu, \nu}(\Phi_\mu, \Phi_\nu) - ia_\mu A_{\mu, \nu}(|\nabla|\Phi_\mu, \Phi_\nu) - ia_\nu A_{\mu, \nu}(\Phi_\mu, |\nabla|\Phi_\nu) = 0, \tag{3.13}$$

which gives us,

$$a_{\mu, \nu}(\xi - \eta, \eta) = \frac{i}{(|\xi| - a_\mu |\xi - \eta| - a_\nu |\eta|)} \tilde{m}'_{\mu, \nu}(\xi - \eta, \eta). \tag{3.14}$$

**3.3. The  $S^\infty$  norm estimate of symbols.** In this section, we first discuss how to estimate the  $S_{k,k_1,k_2}^\infty$  norm for a general symbol which depends on the angular variable, and then we estimate the  $S_{k,k_1,k_2}^\infty$  norm of  $a_{\mu,\nu}(\xi - \eta, \eta)$ ,  $(\mu, \nu) \in \mathcal{S}$ . The method stated here can be easily generalized to the three independent variables setting. One can estimate  $S_{k,k_1,k_2,k_3}^\infty$  norm of a symbol very similarly. For later stated  $S^\infty$  norm estimates of symbols, readers can refer to this subsection for help to see the validities of those stated estimates.

The essential tool we use is Lemma 2.2. Beside the pointwise estimate, we also have to estimate the derivatives of symbols. Since the angular variable of symbol may appear in the denominator, it's not so straightforward to see the upper bound of  $S_{k,k_1,k_2}^\infty$  norm directly. Hence, we provide the following guide of doing estimates to readers, which consists of two essential steps and an example.

**Step 1:** Choose the independent variables.

Since there are only two independent variables among three variables:  $\xi$ ,  $\xi - \eta$ , and  $\eta$ . It's important to choose the right two independent variables, otherwise the estimate can be unnecessarily rough. We choose the least two of the three variables as the independent variables. For example, if  $|\xi| \leq |\xi - \eta| \leq |\eta|$ , then we let  $\xi_1 := \xi$  and  $\xi_2 := \xi - \eta$  to be independent and write  $\eta$  as  $\xi_1 - \xi_2$ .

For a symbol as follows

$$m(\xi, \eta) \psi_k(\xi) \psi_{k_1}(\xi - \eta) \psi_{k_2}(\eta),$$

variables  $\xi$  and  $\eta$  are not always the independent variables. If  $|\xi - \eta| \leq |\xi| \sim |\eta|$ , then we let  $\xi - \eta$  and  $\xi$  to be independent variables first and then apply Lemma 2.2.

**Step 2:** View the angular part as a whole part when we have angular variable in the denominator.

The main point of this step is that we can reformulate the aforementioned symbols as follows,

$$m(\xi, \eta) = \tilde{m}(\xi, \eta) f\left(\frac{\eta}{|\eta|} \times \frac{\xi - \eta}{|\xi - \eta|}, (1, 0) \times \frac{\eta}{|\eta|}\right) = \tilde{m}(\xi, \eta) f(\sin(\angle(\xi - \eta, \eta)), \sin(\angle(\eta, (1, 0)))),$$

where  $m(\xi, \eta)$  is one of the aforementioned symbol,  $\tilde{m}(\xi, \eta)$  is a regular symbol<sup>1</sup> and  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is a smooth function.

**An example:** We choose  $a_{+,+}(\xi - \eta, \eta)$  as a representative example, other symbols can be done similarly. From (3.9) and (3.14), we have

$$\begin{aligned} a_{+,+}(\xi - \eta, \eta) &= \frac{i\tilde{m}'_{+,+}(\xi - \eta, \eta)}{|\xi| - |\xi - \eta| - |\eta|} = \frac{i(|\xi| + |\xi - \eta| + |\eta|)}{2|\xi - \eta||\eta|(\cos(\angle(\xi - \eta, \eta)) - 1)} \tilde{m}'_{+,+}(\xi - \eta, \eta) \\ &= \frac{\xi \cdot (\xi - 2\eta)}{16\pi|\xi|^2} \frac{i(|\xi| + |\xi - \eta| + |\eta|)}{2|\xi - \eta||\eta|} \left( (\xi - \eta) \times \eta \right) + \left( \frac{-i(|\xi| + |\xi - \eta| + |\eta|)}{2(\cos(\angle(\xi - \eta, \eta)) - 1)} \frac{\xi}{16\pi|\xi|} \cdot \left( \frac{\xi - \eta}{|\xi - \eta|} - \frac{\eta}{|\eta|} \right) \right) \times \\ &\quad \left( \frac{(\xi - \eta)}{|\xi - \eta|} \times \frac{\eta}{|\eta|} \right) = \underbrace{\frac{i\xi \cdot (\xi - 2\eta)(|\xi| + |\xi - \eta| + |\eta|)}{32\pi|\xi|^2} \left( \frac{(\xi - \eta)}{|\xi - \eta|} \times \frac{\eta}{|\eta|} \right)}_{\text{Part I: A regular symbol}} \\ &\quad + \underbrace{\frac{-i(|\xi| + |\xi - \eta| + |\eta|)\xi_1}{32\pi|\xi|} \frac{\cos(\angle(\xi - \eta, (1, 0))) - \cos(\angle(\eta, (1, 0)))}{(\cos(\angle(\xi - \eta, \eta)) - 1)} \left( \frac{(\xi - \eta)}{|\xi - \eta|} \times \frac{\eta}{|\eta|} \right)}_{\text{Part II}} \end{aligned}$$

<sup>1</sup>For a regular symbol, after choosing the independent variables as in Step 1, the right hand side of (2.1) is comparable to the  $L^\infty$  norm of itself.

$$+ \underbrace{\frac{-i(|\xi| + |\xi - \eta| + |\eta|)\xi_2}{32\pi|\xi|} \frac{\sin(\angle(\xi - \eta, (1, 0))) - \sin(\angle(\eta, (1, 0)))}{(\cos(\angle(\xi - \eta, \eta)) - 1)} \left( \frac{(\xi - \eta)}{|\xi - \eta|} \times \frac{\eta}{|\eta|} \right)}_{\text{Part III}},$$

where  $\xi_j, j \in \{1, 2\}$ , is the  $j$ -th component of vector  $\xi$  and we used the following fact in above computation,

$$\frac{\xi - \eta}{|\xi - \eta|} - \frac{\eta}{|\eta|} = (\cos(\angle(\xi - \eta, (1, 0))) - \cos(\angle(\eta, (1, 0))), \sin(\angle(\xi - \eta, (1, 0))) - \sin(\angle(\eta, (1, 0)))).$$

It remains to check “Part II” and “Part III”. Using the Step 2, we rewrite them as follows,

$$\begin{aligned} \text{Part II} &= \underbrace{\frac{-i(|\xi| + |\xi - \eta| + |\eta|)\xi_1}{32\pi|\xi|}}_{\text{A regular symbol}} \underbrace{f\left(\frac{\eta}{|\eta|} \times \frac{\xi - \eta}{|\xi - \eta|}, (1, 0) \times \frac{\eta}{|\eta|}\right)}_{\text{Angular part}}, \\ \text{Part III} &= \underbrace{\frac{-i(|\xi| + |\xi - \eta| + |\eta|)\xi_2}{32\pi|\xi|}}_{\text{A regular symbol}} \underbrace{g\left(\frac{\eta}{|\eta|} \times \frac{\xi - \eta}{|\xi - \eta|}, (1, 0) \times \frac{\eta}{|\eta|}\right)}_{\text{Angular part}}, \end{aligned}$$

where

$$\begin{aligned} f(x, y) &= \frac{-x(\cos(\sin^{-1}(y) + \sin^{-1}(x)) - \cos(\sin^{-1}(y)))}{(\cos(\sin^{-1}(x)) - 1)}, \\ g(x, y) &= \frac{-x(\sin(\sin^{-1}(y) + \sin^{-1}(x)) - y)}{(\cos(\sin^{-1}(x)) - 1)}, \end{aligned}$$

and  $x, y \in [-1, 1]$ . We have the following expansions when  $x, y$  are very close to 0,

$$\begin{aligned} f(x, y) &= x + 2y - \frac{y^2x + x^2y}{2} + o(x^4) + o(y^4), \quad \text{when } |x|, |y| \ll 1, \\ g(x, y) &= -2 + y^2 + xy + \frac{4x^2 - 2x^2y^2 + x^4}{8} + \mathcal{O}(x^4) + \mathcal{O}(y^4), \quad \text{when } |x|, |y| \ll 1. \end{aligned}$$

We first use the rules in Step 1 to find out the independent variables and then use the Chain rule and Leibniz's rule, as a result, we can see that the “Angular parts” of “Part II” and “Part III” are also regular symbols.

From above discussion, it's easy to see the following estimate holds

$$|a_{+,+}(\xi - \eta, \eta)\psi_k(\xi)\psi_{k_1}(\xi - \eta)\psi_{k_2}(\eta)| \lesssim 2^{\max\{k_1, k_2\}},$$

and then one can check the derivatives of  $a_{+,+}(\xi - \eta, \eta)$  with respect to the independent variables, from Lemma 2.2, eventually the following estimate holds,

$$\|a_{+,+}(\xi - \eta, \eta)\|_{\mathcal{S}_{k, k_1, k_2}^\infty} \lesssim 2^{\max\{k_1, k_2\}}.$$

We can perform similar analysis for the other two cases:  $a_{+,-}(\xi - \eta, \eta)$  and  $a_{-,-}(\xi - \eta, \eta)$  and have the following lemma,

**Lemma 3.1.** *For any admissible  $k, k_1, k_2 \in \mathbb{Z}$ , we have the following estimate for any  $(\mu, \nu) \in \mathcal{S}$ ,*

$$\|a_{\mu, \nu}(\xi - \eta, \eta)\|_{\mathcal{S}_{k, k_1, k_2}^\infty} \lesssim 2^{\max\{k_1, k_2\}}. \quad (3.15)$$

*Proof.* The desired estimate (3.15) follows from above discussion.  $\square$

**3.4. The usual energy estimate.** To find out what cubic correction terms to add, we first do the usual energy estimate. Due to the quasilinear nature of the system (1.19), we have to avoid losing derivatives when doing energy estimate.

**3.4.1. Energy in terms of  $\phi_0$  and  $\Phi$ .** We define the usual energy as follows,

$$E(t) := E^{N_0}(t) + E^{N_1}(t) + E^0(t), \quad E^{N_0}(t) := \sum_{k+j=N_0, 0 \leq k, j \in \mathbb{Z}} \frac{1}{2} \left[ \int_{\mathbb{R}^2} |\partial_1^k \partial_2^j \phi_0|^2 + |\partial_1^k \partial_2^j \Phi|^2 \right], \quad (3.16)$$

$$E^{N_1}(t) := \sum_{k+j=N_1, 0 \leq k, j \in \mathbb{Z}} \frac{1}{2} \left[ \int_{\mathbb{R}^2} |\partial_1^k \partial_2^j S \phi_0|^2 + |\partial_1^k \partial_2^j S \Phi|^2 + |\partial_1^k \partial_2^j \Omega \phi_0|^2 + |\partial_1^k \partial_2^j \Omega \Phi|^2 \right],$$

$$E^0(t) := \frac{1}{2} \left[ \int_{\mathbb{R}^2} |\phi_0|^2 + |\Phi|^2 + |S \phi_0|^2 + |\Omega \phi_0|^2 + |S \Phi|^2 + |\Omega \Phi|^2 \right].$$

For a tuple of nonnegative integers  $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ ,  $|\alpha| > 0$  and  $|\alpha_3| + |\alpha_4| \leq 1$ , we use  $\Gamma^\alpha$  to denote  $\partial_1^{\alpha_1} \partial_2^{\alpha_2} S^{\alpha_3} \Omega^{\alpha_4}$  and use  $f^\alpha$  to denote  $\Gamma^\alpha f$  for a well defined function  $f$ . For a bilinear term  $T(f, g)$ , we use

$$T^{(\beta, \gamma)}(f, g) := T(f^\beta, g^\gamma) + T(f^\gamma, g^\beta),$$

to denote the terms that one of the inputs hit by  $\Gamma^\beta$  and the other one hit by  $\Gamma^\gamma$ . As

$$[S, \partial_t + i|\nabla|] = -(\partial_t + i|\nabla|), \quad [\Omega, \partial_t + i|\nabla|] = 0, \quad (3.17)$$

after applying  $\Gamma^\alpha$ ,  $\Gamma^\alpha S$ , and  $\Gamma^\alpha \Omega$  to the system of equations (1.19), we can derive equations satisfied by  $\phi_0^\alpha \in \{\Gamma^\alpha \phi_0, \Gamma^\alpha S \phi_0, \Gamma^\alpha \Omega \phi_0\}$  and  $\Phi^\alpha \in \{\Gamma^\alpha \Phi, \Gamma^\alpha S \Phi, \Gamma^\alpha \Omega \Phi\}$  as follows,

$$\begin{cases} \partial_t \phi_0^\alpha = \mathcal{N}_0^{(\alpha, 0)} + \mathbf{Err}_0^\alpha \\ \partial_t \Phi^\alpha + i|\nabla| \Phi^\alpha = \mathcal{N}_1^{(\alpha, 0)} + \mathbf{Err}_1^\alpha, \end{cases} \quad (3.18)$$

where  $\mathbf{Err}_0^\alpha$  and  $\mathbf{Err}_1^\alpha$  are good error terms, which consist of terms in which two inputs are not hit by the entire  $\Gamma^\alpha$  derivative and the commutator terms if “ $S$ ” or “ $\Omega$ ” is applied. More precisely, for  $i \in \{0, 1\}$ ,

$$\mathbf{Err}_i^\alpha = \sum_{|\gamma| \leq |\beta| < |\alpha|, \beta + \gamma = \alpha} \binom{\alpha}{\beta} \mathcal{N}_i^{(\beta, \gamma)} + \text{commutator terms if } |\alpha_3| + |\alpha_4| = 1, \quad \binom{\alpha}{\beta} := \prod_{j=1}^4 \binom{\alpha_j}{\beta_j}. \quad (3.19)$$

We mention that we need to utilize the commutation rules for the vector fields to derive the system (3.18). For readers’ conveniences, we derive and discuss those commutations rules before ending this subsubsection.

For two smooth well defined functions  $h_1$  and  $h_2$  and a bilinear operator  $Q(\cdot, \cdot)$  with symbol  $q(\xi, \eta)$ , which is homogeneous of degree  $c$ , i.e.,

$$q(\lambda \xi, \lambda \eta) = \lambda^c q(\xi, \eta), \quad (3.20)$$

we have

$$SQ(h_1, h_2) = Q(S h_1, h_2) + Q(h_1, S h_2) - c Q(h_1, h_2), \quad c \in \{1, 2\}. \quad (3.21)$$

To understand why (3.21) holds, it is sufficient to consider the “ $x \cdot \nabla$ ” part of scaling vector “ $S$ ” as “ $t \partial_t$ ” part of “ $S$ ” distributes as usual derivatives. We have

$$\mathcal{F}(x \cdot \nabla Q(h_1, h_2))(\xi) = [-\xi \cdot \nabla_\xi - 2I] \left( \int_{\mathbb{R}^2} q(\xi, \eta) \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta \right)$$

$$\begin{aligned}
&= \int_{\mathbb{R}^2} [(-\xi \cdot \nabla_\xi - 2)q(\xi, \eta)] \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta + \\
&\quad \int_{\mathbb{R}^2} q(\xi, \eta) \eta \cdot \nabla_\eta \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta - \int_{\mathbb{R}^2} q(\xi, \eta) (\xi - \eta) \cdot \nabla_\xi \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta \\
&= \int_{\mathbb{R}^2} q(\xi, \eta) [-(\xi - \eta) \cdot \nabla_\xi - 2] \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta + \int_{\mathbb{R}^2} q(\xi, \eta) \widehat{h}_1(\xi - \eta) [-\eta \cdot \nabla_\eta - 2] \widehat{h}_2(\eta) d\eta \\
&\quad - \int_{\mathbb{R}^2} (\xi \cdot \nabla_\xi + \eta \cdot \nabla_\eta) q(\xi, \eta) \widehat{h}_1(\xi - \eta) \widehat{h}_2(\eta) d\eta. \tag{3.22}
\end{aligned}$$

After taking one derivative with respect to  $\lambda$  for identity (3.20) and evaluating it at  $\lambda = 1$ , we have

$$\xi \cdot \nabla_\xi q(\xi, \eta) + \eta \cdot \nabla_\eta q(\xi, \eta) = cq(\xi, \eta). \tag{3.23}$$

Therefore,

$$x \cdot \nabla Q(h_1, h_2) = Q(x \cdot \nabla h_1, h_2) + Q(h_1, x \cdot \nabla h_2) - cQ(h_1, h_2), \tag{3.24}$$

which implies that (3.21) holds.

Very similarly, we have the following identity for the rotational vector field,

$$\Omega Q(h_1, h_2) = Q(\Omega h_1, h_2) + Q(h_1, \Omega h_2) - Q'(h_1, h_2), \tag{3.25}$$

where bilinear operator  $Q'(\cdot, \cdot)$  is defined by the following symbol,

$$q'(\xi, \eta) = \xi^\perp \cdot \nabla_\xi q(\xi, \eta) + \eta^\perp \cdot \nabla_\eta q(\xi, \eta). \tag{3.26}$$

The size of symbol  $q'(\xi, \eta)$  is comparable to size of  $q(\xi, \eta)$ . Moreover,  $q'(\xi, \eta)$  has null structure as long as  $q(\xi, \eta)$  has null structure. To see this point, we only have to check the case when both  $\nabla_\xi$  and  $\nabla_\eta$  hits the angular part, for example,  $(\xi - \eta) \times \eta$ , we have

$$\xi^\perp \cdot \nabla_\xi ((\xi - \eta) \times \eta) + \eta^\perp \cdot \nabla_\eta ((\xi - \eta) \times \eta) = \xi \cdot \eta - \xi \cdot \eta = 0, \tag{3.27}$$

which infers that bilinear operator  $Q'(\cdot, \cdot)$  also has two degrees of angle inside.

From (3.21) and (3.25), we know that, modulo the good error commutator terms, we can distribute the vector fields  $S$  and  $\Omega$  as usual derivatives.

We remark that commutator terms come from two sources: (i) from the commutation rules in (3.17); (ii) from the commutation rules in (3.21) and (3.25). It is safe to put those commutator terms into the error terms, because the commutator terms only depend on the  $\phi_0$  and  $\Phi$  and their top regularities are all  $N_0$ , which is much bigger than  $N_1$ .

**3.4.2. The usual energy estimate.** Recall the definition of  $E^{N_0}(t)$  and the system of equations (3.18), we have

$$\frac{d}{dt} E^{N_0}(t) = \sum_{|\alpha|=N_0, \alpha_3=\alpha_4=0} \operatorname{Re} \left( \int \overline{\phi_0^\alpha} \mathbf{Err}_0^\alpha + \overline{\Phi^\alpha} \mathbf{Err}_1^\alpha \right) + \operatorname{Re} \left( \int \overline{\phi_0^\alpha} \mathcal{N}_0^{(\alpha,0)} + \overline{\Phi^\alpha} \mathcal{N}_1^{(\alpha,0)} \right). \tag{3.28}$$

From the system of equation satisfied by  $\psi$ ,  $G_1$ , and  $G_2$  in (1.15), we have

$$\mathcal{N}_0^{(\alpha,0)} = R_1 \widetilde{\mathcal{N}}_2^{(\alpha,0)} - R_2 \widetilde{\mathcal{N}}_1^{(\alpha,0)}, \quad \mathcal{N}_1^{(\alpha,0)} = \widetilde{\mathcal{N}}_0^{(\alpha,0)} + i[R_1 \widetilde{\mathcal{N}}_1^{(\alpha,0)} + R_2 \widetilde{\mathcal{N}}_2^{(\alpha,0)}].$$



From (1.15), we can replace  $\phi_0$  and  $\Phi$  in the second integral of (3.28) by  $\psi$ ,  $G_1$ , and  $G_2$ , which are all real. As a result, we have

$$\operatorname{Re}\left(\int \overline{\phi_0^\alpha} \mathcal{N}_0^{(\alpha,0)} + \overline{\Phi^\alpha} \mathcal{N}_1^{(\alpha,0)}\right) = \int \psi^\alpha \tilde{\mathcal{N}}_0^{(\alpha,0)} + G_1^\alpha \tilde{\mathcal{N}}_1^{(\alpha,0)} + G_2^\alpha \tilde{\mathcal{N}}_2^{(\alpha,0)}. \quad (3.29)$$

Recall (1.10), we have

$$\tilde{\mathcal{N}}_0 = Q(\psi, \psi) - Q(G_1, G_1) - Q(G_2, G_2), \quad \tilde{\mathcal{N}}_1 = Q_{1,2}(G_1, \psi), \quad \tilde{\mathcal{N}}_2 = Q_{1,2}(G_2, \psi), \quad (3.30)$$

where bilinear operator  $Q(\cdot, \cdot)$  is defined by the symbol as follows,

$$q(\xi - \eta, \eta) = \frac{\xi \cdot (\xi - \eta)}{2\pi|\xi|^2}(\xi - \eta) \times \eta.$$

After utilizing symmetries on the Fourier side, we have

$$\begin{aligned} (3.29) &= \int \psi^\alpha [Q(\psi^\alpha, \psi) + Q(\psi, \psi^\alpha)] + \sum_{i=1,2} G_i^\alpha Q_{1,2}(G_i^\alpha, \psi) + \psi^\alpha [-Q(G_i^\alpha, G_i) - Q(G_i, G_i^\alpha)] \\ &+ G_i^\alpha Q_{1,2}(G_i, \psi^\alpha) = \sum_{i=1,2} \int \int \left[ \overline{\widehat{G_i^\alpha}(\xi)} \widehat{G_i^\alpha}(\xi - \eta) \widehat{\psi}(\eta) q_1(\xi - \eta, \eta) + \overline{\widehat{\psi^\alpha}(\xi)} \widehat{G_i^\alpha}(\xi - \eta) \widehat{G_i}(\eta) q_2(\xi - \eta) \right] d\xi d\eta \\ &+ \int \int \overline{\widehat{\psi^\alpha}(\xi)} \widehat{\psi^\alpha}(\xi - \eta) \widehat{\psi}(\eta) q_3(\xi - \eta, \eta) d\xi d\eta, \end{aligned} \quad (3.31)$$

where

$$q_1(\xi - \eta, \eta) = \left[ \frac{1}{2\pi}(\xi - \eta) \times (\eta) + \frac{1}{2\pi}\xi \times (-\eta) \right] / 2 = 0,$$

$$\begin{aligned} q_2(\xi - \eta, \eta) &= \frac{1}{2\pi}(-\eta) \times \xi - q(\xi - \eta, \eta) - q(\eta, \xi - \eta) \\ &= \frac{1}{2\pi}(\xi - \eta) \times \eta - \frac{\xi \cdot (\xi - 2\eta)}{2\pi|\xi|^2}(\xi - \eta) \times \eta = \frac{\xi \cdot \eta}{\pi|\xi|^2}(\xi - \eta) \times \eta, \end{aligned} \quad (3.32)$$

$$\begin{aligned} q_3(\xi - \eta, \eta) &= [q(\xi - \eta, \eta) + q(\eta, \xi - \eta)] / 2 + [q(\xi, -\eta) + q(-\eta, \xi)] / 2 \\ &= -\left( \frac{\xi \cdot \eta}{2\pi|\xi|^2} + \frac{(\xi - \eta) \cdot \eta}{2\pi|\xi - \eta|^2} \right) (\xi - \eta) \times \eta = -(q_2(\xi - \eta, \eta) + q_2(\xi, -\eta)) / 2. \end{aligned} \quad (3.33)$$

By the Lemma 2.2, it's easy to see the following estimate holds from above explicit formulas,

$$\|q_2(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} + \|q_3(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{2k_2}, \quad \text{if } k_2 \leq k_1 - 10. \quad (3.34)$$

For (3.29), we only have to worry about losing derivative when  $|\eta| \ll |\xi - \eta|$ , i.e.,  $k_2 \leq k_1 - 10$ , from estimate (3.34), we can see that it is of semilinear type and doesn't lose derivative.

After replacing  $\psi$ ,  $G_1$ , and  $G_2$  by  $\phi_0$  and  $\Phi$  through (1.16), we have

$$\begin{aligned} (3.29) &= \int \sum_{i=1,2} \psi^\alpha Q_2(G_i^\alpha, G_i) + \psi^\alpha Q_3(\psi^\alpha, \psi) = \operatorname{Re}\left( \int \sum_{\mu,\nu,\kappa} \frac{1}{8} \Phi_\mu^\alpha Q_3(\Phi_\nu^\alpha, \Phi_\kappa) + \sum_{i=1,2} \frac{1}{8} (\Phi^\alpha \right. \\ &\left. + \overline{\Phi^\alpha}) Q_2(\Gamma^\alpha((-1)^i R_{3-i} \phi_0 + c_+ R_i \Phi + c_- R_i \overline{\Phi}), (-1)^i R_{3-i} \phi_0 + c_+ R_i \Phi + c_- R_i \overline{\Phi}) \right), \end{aligned}$$

where bilinear operators  $Q_2(\cdot, \cdot)$  and  $Q_3(\cdot, \cdot)$  are defined by the symbols  $q_2(\cdot, \cdot)$  and  $q_3(\cdot, \cdot)$  respectively. To sum up, we have

$$\begin{aligned} \frac{d}{dt}E^{N_0}(t) &= \sum_{\substack{|\alpha|=N_0 \\ \alpha_3=\alpha_4=0}} \operatorname{Re} \left( \int \overline{\phi_0^\alpha} \mathbf{Err}_0^\alpha + \overline{\Phi}^\alpha \mathbf{Err}_1^\alpha \right) + \operatorname{Re} \left( \int \sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \Phi_\mu^\alpha Q_3(\Phi_\nu^\alpha, \Phi_\kappa) \right. \\ &+ \sum_{i=1,2} \frac{1}{8} (\Phi^\alpha + \overline{\Phi}^\alpha) Q_2(\Gamma^\alpha((-1)^i 2R_{3-i}\phi_0 + c_+ R_i \Phi + c_- R_i \overline{\Phi}), (-1)^i 2R_{3-i}\phi_0 + c_+ R_i \Phi + c_- R_i \overline{\Phi}) \Big). \end{aligned} \quad (3.35)$$

With minor modifications, we can perform the same procedure for  $E^{N_1}(t)$  and have the following,

$$\begin{aligned} \frac{d}{dt}E^{N_1}(t) &= \sum_{\substack{|\alpha|=N_1+1 \\ \alpha_3+\alpha_4=1}} \operatorname{Re} \left( \int \overline{\phi_0^\alpha} \mathbf{Err}_0^\alpha + \overline{\Phi}^\alpha \mathbf{Err}_1^\alpha \right) + \operatorname{Re} \left( \int \sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \Phi_\mu^\alpha Q_3(\Phi_\nu^\alpha, \Phi_\kappa) \right. \\ &+ \sum_{i=1,2} \frac{1}{8} (\Phi^\alpha + \overline{\Phi}^\alpha) Q_2(\Gamma^\alpha((-1)^i 2R_{3-i}\phi_0 + c_+ R_i \Phi + c_- R_i \overline{\Phi}), (-1)^i 2R_{3-i}\phi_0 + c_+ R_i \Phi + c_- R_i \overline{\Phi}) \Big). \end{aligned} \quad (3.36)$$

**3.5. Identifying the most problematic terms.** The most problematic terms are the “cubic” terms (recall that we called  $\phi_0$  itself *quadratic and higher* in the sense of decay) inside the derivative of energy  $E(t)$ , which decay very slowly and only depend on  $\Phi$ . We identify them in this section. Recall the definition of  $E(t)$  in (3.16), we have

$$\begin{aligned} \text{“cubic” terms of } \frac{d}{dt}E(t) &= \text{cubic part of } \frac{d}{dt}(E^0(t) + E^{N_0}(t) + E^{N_1}(t)), \\ \text{cubic part of } \frac{d}{dt}E^0(t) &= \sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int \overline{\Phi} \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu) + \overline{\Omega \Phi} [\tilde{Q}_{\mu, \nu}(\Omega \Phi_\mu, \Phi_\nu) + \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Omega \Phi_\nu) \right. \\ &\quad \left. - \tilde{Q}'_{\mu, \nu}(\Phi_\mu, \Phi_\nu)] + \overline{S \Phi} [\tilde{Q}_{\mu, \nu}(S \Phi_\mu, \Phi_\nu) + \tilde{Q}_{\mu, \nu}(\Phi_\mu, S \Phi_\nu) - \tilde{Q}_{\mu, \nu}(\Phi_\mu, \Phi_\nu)] \right), \end{aligned} \quad (3.37)$$

where the bilinear operator  $\tilde{Q}'_{\mu, \nu}(\cdot, \cdot)$ , follows from (3.26), is defined by the following symbol

$$(\xi^\perp \cdot \nabla_\xi + \eta^\perp \cdot \nabla_\eta) \tilde{m}'_{\mu, \nu}(\xi - \eta, \eta).$$

From (3.35), we have

$$\begin{aligned} \text{“cubic” terms of } \frac{d}{dt}E^{N_0}(t) &= \sum_{|\alpha|=N_0, \alpha_3=\alpha_4=0} \sum_{|\gamma| \leq |\beta| < |\alpha|, \beta+\gamma=\alpha} \binom{\alpha}{\beta} \sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int \overline{\Phi}^\alpha \tilde{Q}_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right) \\ &+ \sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \operatorname{Re} \left( \int \sum_{i=1,2} c_\nu c_\kappa \Phi_\mu^\alpha Q_2(R_i \Phi_\nu^\alpha, R_i \Phi_\kappa) + \Phi_\mu^\alpha Q_3(\Phi_\nu^\alpha, \Phi_\kappa) \right). \end{aligned}$$

To see structures inside the second part of above equation, an important step here is to utilize the symmetries and using the fact that the identity (3.33) holds. As a result, we have

$$\sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \operatorname{Re} \left( \int \sum_{i=1,2} c_\nu c_\kappa \Phi_\mu^\alpha Q_2(R_i \Phi_\nu^\alpha, R_i \Phi_\kappa) + \Phi_\mu^\alpha Q_3(\Phi_\nu^\alpha, \Phi_\kappa) \right)$$

$$\begin{aligned}
&= \sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \operatorname{Re} \left( \int \sum_{i=1,2} c_\nu c_\kappa \Phi_\mu^\alpha Q_2(R_i \Phi_\nu^\alpha, R_i \Phi_\kappa) - \Phi_\mu^\alpha Q_2(\Phi_\nu^\alpha, \Phi_\kappa)/2 - \Phi_\nu^\alpha Q_2(\Phi_\mu^\alpha, \Phi_{-\kappa})/2 \right) \\
&= \sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \operatorname{Re} \left( \int \sum_{i=1,2} c_\nu c_\kappa \Phi_\mu^\alpha Q_2(R_i \Phi_\nu^\alpha, R_i \Phi_\kappa) - \Phi_\mu^\alpha Q_2(\Phi_\nu^\alpha, \Phi_\kappa) \right) \\
&= \sum_{\mu, \nu, \kappa \in \{+, -\}} \frac{1}{8} \operatorname{Re} \left( \int \Phi_\mu^\alpha Q_{\nu, \kappa}^4(\Phi_\nu^\alpha, \Phi_\kappa) \right) = \sum_{\nu, \kappa \in \{+, -\}} \frac{1}{4} \operatorname{Re} \left( \int \overline{\Phi}^\alpha Q_{\nu, \kappa}^4(\Phi_\nu^\alpha, \Phi_\kappa) \right), \quad (3.38)
\end{aligned}$$

where the bilinear operator  $Q_4(\cdot, \cdot)$  is defined by the following symbol,

$$q_{\nu, \kappa}^4(\xi - \eta, \eta) = -q_2(\xi - \eta, \eta) \left( 1 + c_\nu c_\kappa \frac{(\xi - \eta) \cdot \eta}{|\xi - \eta| |\eta|} \right) = -q_2(\xi - \eta, \eta) (1 - a_\nu a_\kappa \cos(\angle(\xi - \eta, \eta))). \quad (3.39)$$

To sum up, we have

$$\begin{aligned}
\text{“cubic” terms of } \frac{d}{dt} E^{N_0}(t) &= \sum_{\substack{|\alpha|=N_0 \\ \alpha_3=\alpha_4=0}} \sum_{\substack{|\beta|, |\gamma| < |\alpha| \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int \overline{\Phi}^\alpha \tilde{Q}_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right) \\
&+ \sum_{\nu, \kappa \in \{+, -\}} \frac{1}{4} \operatorname{Re} \left( \int \overline{\Phi}^\alpha Q_{\nu, \kappa}^4(\Phi_\nu^\alpha, \Phi_\kappa) \right). \quad (3.40)
\end{aligned}$$

Now we consider the cubic part of the derivative of  $E^{N_1}(t)$ . Except the value of  $\alpha$  is different and the presence of commutators terms, most cubic terms are same as terms in (3.40). Very similarly, we have

$$\begin{aligned}
\text{“cubic” terms of } \frac{d}{dt} E^{N_1}(t) &= \sum_{\substack{|\alpha|=N_1+1 \\ \alpha_3+\alpha_4=1}} \left[ \sum_{\substack{|\gamma|, |\beta| < |\alpha| \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int \overline{\Phi}^\alpha \tilde{Q}_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right) \right. \\
&+ \sum_{\nu, \kappa \in \{+, -\}} \frac{1}{4} \operatorname{Re} \left( \int \overline{\Phi}^\alpha Q_{\nu, \kappa}^4(\Phi_\nu^\alpha, \Phi_\kappa) \right) \Big] + \sum_{\substack{|\alpha|=N_1 \\ \alpha_3=\alpha_4=0}} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int -\overline{\Gamma}^\alpha S \tilde{\Phi} \tilde{Q}_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right. \\
&\quad \left. - \overline{\Gamma}^\alpha \Omega \tilde{\Phi} \tilde{Q}'_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right). \quad (3.41)
\end{aligned}$$

To sum up, from (3.37), (3.40) and (3.41), we have

$$\begin{aligned}
\text{“cubic” terms of } \frac{d}{dt} E(t) &= \sum_{\substack{(|\alpha_1|+|\alpha_2|, |\alpha_3|+|\alpha_4|) \in \\ \{(N_0, 0), (N_1, 1), (0, 0), (0, 1)\}}} \sum_{\substack{|\beta|, |\gamma| \leq \max\{|\alpha|-1, 1\} \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \times \\
&\sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int \overline{\Phi}^\alpha \tilde{Q}_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right) + \sum_{\substack{(|\alpha_1|+|\alpha_2|, |\alpha_3|+|\alpha_4|) \in \\ \{(N_0, 0), (N_1, 1)\}}} \sum_{\nu, \kappa \in \{+, -\}} \frac{1}{4} \operatorname{Re} \left( \int \overline{\Phi}^\alpha Q_{\nu, \kappa}^4(\Phi_\nu^\alpha, \Phi_\kappa) \right) \\
&+ \sum_{\substack{|\alpha| \in \{0, N_1\} \\ \alpha_3=\alpha_4=0}} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \sum_{(\mu, \nu) \in \mathcal{S}} \operatorname{Re} \left( \int -\overline{\Gamma}^\alpha S \tilde{\Phi} \tilde{Q}_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) - \overline{\Gamma}^\alpha \Omega \tilde{\Phi} \tilde{Q}'_{\mu, \nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right). \quad (3.42)
\end{aligned}$$

### 3.6. Construction of the modified energy.

**3.6.1. First level correction.** We modify the usual energy by adding the cubic correction terms, which cancels out the problematic cubic terms in (3.42). From the construction of normal form transformation in subsection 3.2, to cancel out (3.42), we define

$$\begin{aligned}
E_{FCorr}(t) = & \sum_{\substack{(|\alpha_1|+|\alpha_2|,|\alpha_3|+|\alpha_4|) \in \\ \{(N_0,0),(N_1,1),(0,0),(0,1)\}}} \sum_{\substack{|\beta|,|\gamma| \leq (|\alpha|-1)_+ \\ \beta+\gamma=\alpha}} \binom{\alpha}{\beta} \sum_{(\mu,\nu) \in \mathcal{S}} \operatorname{Re} \left( \int \overline{\Phi}^\alpha A_{\mu,\nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right) \\
& + \sum_{\substack{(|\alpha_1|+|\alpha_2|,|\alpha_3|+|\alpha_4|) \in \\ \{(N_0,0),(N_1,1)\}}} \sum_{\nu,\kappa \in \{+,-\}} \frac{1}{4} \operatorname{Re} \left( \int \overline{\Phi}^\alpha A_{\nu,\kappa}^4(\Phi_\nu^\alpha, \Phi_\kappa) \right) \\
& + \sum_{\substack{|\alpha| \in \{0,N_1\} \\ \alpha_3=\alpha_4=0}} \sum_{\beta+\gamma=\alpha} \binom{\alpha}{\beta} \sum_{(\mu,\nu) \in \mathcal{S}} \operatorname{Re} \left( \int -\overline{\Gamma}^\alpha S \overline{\Phi} A_{\mu,\nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) - \overline{\Gamma}^\alpha \Omega \overline{\Phi} A'_{\mu,\nu}(\Phi_\mu^\beta, \Phi_\nu^\gamma) \right), \quad (3.43)
\end{aligned}$$

where the bilinear operator  $A'_{\mu,\nu}(\cdot, \cdot)$  and the bilinear operator  $A_{\nu,\kappa}^4(\cdot, \cdot)$  are defined by the symbol  $a'_{\mu,\nu}(\cdot, \cdot)$  and  $a_{\nu,\kappa}^4(\xi - \eta, \eta)$  as follows,

$$\begin{aligned}
a'_{\mu,\nu}(\xi - \eta, \eta) &= \frac{i(\xi^\perp \cdot \nabla_\xi + \eta^\perp \cdot \nabla_\eta) \tilde{m}'_{\mu,\nu}(\xi - \eta, \eta)}{|\xi| - a_\mu |\xi - \eta| - a_\nu |\eta|}, \\
a_{\nu,\kappa}^4(\xi - \eta, \eta) &= \frac{i q_{\nu,\kappa}^4(\xi - \eta, \eta)}{|\xi| - a_\nu |\xi - \eta| - a_\kappa |\eta|} = \frac{i(|\xi| + a_\nu |\xi - \eta| + a_\kappa |\eta|) q_2(\xi - \eta, \eta)}{2a_\nu a_\kappa |\xi - \eta| |\eta|}, \quad (3.44)
\end{aligned}$$

and  $q_2(\cdot, \cdot)$  is explicitly given in (3.32). As mentioned in the commutation part of the rotational vector field  $\Omega$ , symbol  $(\xi^\perp \cdot \nabla_\xi + \eta^\perp \cdot \nabla_\eta) \tilde{m}'_{\mu,\nu}(\xi - \eta, \eta)$  has two degrees of angle inside, hence  $a'_{\mu,\nu}(\xi - \eta, \eta)$  is a regular symbol.

Similar to the proof of Lemma 3.1, by Lemma 2.2, we can derive the following estimates,

$$\|a'_{\mu,\nu}(\xi - \eta, \eta)\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{\max\{k_1,k_2\}}, \quad \|a_{\nu,\kappa}^4(\xi - \eta, \eta)\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{k_2}. \quad (3.45)$$

**3.6.2. Second level correction.** Due to the quasilinear nature, there are potentials of losing a derivative after taking derivative with respect to time  $t$  for  $E_{FCorr}(t)$ . We first identify those problematic terms and then figure out how to get around this difficulty.

We first claim that the following terms are most problematic and then explain reasons,

$$\begin{aligned}
\text{Problematic terms} := & \sum_{\substack{(|\alpha_1|+|\alpha_2|,|\alpha_3|+|\alpha_4|) \in \\ \{(N_0,0),(N_1,1)\}}} \sum_{\substack{\beta+\gamma=\alpha, |\gamma|=1, \\ |\gamma_3|=|\gamma_4|=0}} \binom{\alpha}{\beta} \operatorname{Re} \left( \int \overline{\Gamma}^\gamma \overline{\Phi}^\beta (A_{+,+}(\Phi^\beta, \Phi^\gamma) + A_{+,+}(\Phi^\gamma, \Phi^\beta) \right. \\
& \left. + A_{+,-}(\Phi^\beta, \overline{\Phi}^\gamma) \right) + \frac{1}{4} \operatorname{Re} \left( \int \overline{\Phi}^\alpha (A_{+,+}^4(\Phi^\alpha, \Phi) + A_{+,-}^4(\Phi^\alpha, \overline{\Phi})) \right). \quad (3.46)
\end{aligned}$$

To see why aforementioned terms are the most problematic terms, one only needs to check which terms are possible to lose one derivative. Since  $A_{\mu,\nu}(\cdot, \cdot)$  only loses at most one derivative, one of two inputs has to have the allowed maximal derivatives to lose one derivative. Moreover, we only have to worry about losing a derivative when the frequency of  $\Phi^\beta$  (or  $\Phi^\alpha$ ) is larger than  $\Phi^\gamma$  (or  $\Phi_\pm$ ). In this scenario, we have better estimates for the bilinear operators associated with the  $\overline{\Phi}^\beta \overline{\Phi}^\beta (\overline{\Phi}^\alpha \overline{\Phi}^\alpha)$  type terms. As a result, those type terms do not lose a derivative. That's why we only have  $\overline{\Phi}^\beta \overline{\Phi}^\beta (\overline{\Phi}^\alpha \Phi^\alpha)$  type in (3.46). We will see more details and understand those points clearer in the proof Lemma 3.6.

To avoid losing a derivative after taking a derivative with respect to time for “Problematic terms”, we need to utilize symmetries as what we did in (3.29). It motivates us to define the following second level correction terms,

$$E_{SCorr}(t) := \sum_{\substack{(|\alpha_1|+|\alpha_2|, |\alpha_3|+|\alpha_4|) \in \{(N_0, 0), (N_1, 1)\} \\ \beta+\gamma=\alpha, |\gamma|=1, \\ |\gamma_3|=|\gamma_4|=0}} \sum_{\substack{(\alpha) \\ (\beta)}} \text{Re} \left( \int \overline{\Gamma^\gamma \phi_0^\beta} (A_{+,+}(\phi_0^\beta, \Phi^\gamma) + A_{+,+}(\Phi^\gamma, \phi_0^\beta) \right. \\ \left. + A_{+,-}(\phi_0^\beta, \overline{\Phi^\gamma}) \right) + \frac{1}{4} \text{Re} \left( \int \overline{\phi_0^\alpha} (A_{+,+}^4(\phi_0^\alpha, \Phi) + A_{+,-}^4(\phi_0^\alpha, \overline{\Phi})) \right). \quad (3.47)$$

To sum up, the modified energy that will be used to do energy estimate is defined as follows,

$$E^{modi}(t) = E(t) + E_{FCorr}(t) + E_{SCorr}(t). \quad (3.48)$$

**Lemma 3.2.** *Under the bootstrap assumption (2.2), we have*

$$\sup_{t \in [0, T]} |E_{FCorr}(t)| + |E_{SCorr}(t)| \lesssim \epsilon_0^2. \quad (3.49)$$

*Proof.* Let  $h_1$  and  $h_2$  be two well defined functions. From estimate (3.15) and (3.45) and Lemma 2.1, we have the following estimate for a bilinear operator  $A(\cdot, \cdot) \in \{A_{\mu, \nu}(\cdot, \cdot), A'_{\mu, \nu}(\cdot, \cdot)\}$ ,

$$\begin{aligned} \|A(h_1, h_2)\|_{H^s} &\lesssim \left( \sum_{k_1 \leq k_2 - 10} 2^{2sk_2} \|P_{k_2}[A(P_{k_1}h_1, P_{k_2}h_2)]\|_{L^2}^2 \right)^{1/2} + \left( \sum_{k_2 \leq k_1 - 10} 2^{2sk_1} \times \right. \\ &\quad \left. \|P_{k_1}[A(P_{k_1}h_1, P_{k_2}h_2)]\|_{L^2}^2 \right)^{1/2} + \sum_{|k_1 - k_2| \leq 10} \sum_{k \leq k_1 + 20} 2^{sk} \|P_k[A(P_{k_1}h_1, P_{k_2}h_2)]\|_{L^2} \\ &\lesssim \left( \sum_{k_1 \leq k_2 - 10} 2^{2(s+1)k_2} \|P_{k_2}h_2\|_{L^2}^2 \|P_{k_1}h_1\|_{L^\infty}^2 \right)^{1/2} + \left( \sum_{k_2 \leq k_1 - 10} 2^{2(s+1)k_2} \|P_{k_2}h_2\|_{L^\infty}^2 \|P_{k_1}h_1\|_{L^2}^2 \right)^{1/2} \\ &\quad + \|h_2\|_{H^{s+1}} \|h_1\|_{W^1} \lesssim \|h_2\|_{H^{s+1}} \|h_1\|_{W^1} + \|h_1\|_{H^{s+1}} \|h_2\|_{W^1}. \end{aligned} \quad (3.50)$$

For fixed  $\alpha$  such that  $(|\alpha_1| + |\alpha_2|, |\alpha_3| + |\alpha_4|) = (N_0, 0)$ , then from estimate (3.45) and Lemma 2.2, we have

$$\begin{aligned} \|A_{\mu, \nu}^4(\Phi_\nu^\alpha, \Phi_\kappa)\|_{L^2} &\lesssim \left( \sum_{k_1 \leq k_2 - 10} 2^{2N_0k_1 + 2k_2} \|P_{k_2}\Phi\|_{L^2}^2 \|P_{k_1}\Phi\|_{L^\infty}^2 \right)^{1/2} \\ &+ \left( \sum_{k_2 \leq k_1 - 10} 2^{2(N_0+1)k_2} \|P_{k_2}\Phi\|_{L^2}^2 \|P_{k_1}\Phi\|_{L^\infty}^2 \right)^{1/2} + \|\Phi\|_{H^{N_0}} \|\Phi\|_{W^1} \lesssim \|\Phi\|_{H^{N_0}} \|\Phi\|_{Z'}. \end{aligned} \quad (3.51)$$

Very similarly, we have the following estimate when  $(|\alpha_1| + |\alpha_2|, |\alpha_3| + |\alpha_4|) = (N_1, 1)$ ,

$$\begin{aligned} &\|A_{+, \kappa}^4(\phi_0^\alpha, \Phi_\kappa)\|_{L^2} + \|A_{\mu, \nu}^4(\Phi_\nu^\alpha, \Phi_\kappa)\|_{L^2} \\ &\lesssim (\|S(\Phi, \phi_0)\|_{H^{N_1}} + \|\Omega(\Phi, \phi_0)\|_{H^{N_1}}) \|\Phi\|_{W^{N_1+2}} \lesssim \|\Phi\|_{X_{N_0}} \|\Phi\|_{Z'}. \end{aligned} \quad (3.52)$$

To sum up, from estimates (3.50), (3.51) and (3.52), we can estimate terms inside (3.43) and (3.47) one by one. As a result, we have

$$|E_{FCorr}(t)| + |E_{SCorr}(t)| \lesssim \|(\phi_0, \Phi)\|_{X_{N_0}}^2 \|\Phi\|_{Z'} \lesssim (1+t)^{-1/2+2p_0} \epsilon_1^3 \lesssim \epsilon_0^2. \quad (3.53)$$

Therefore, our desired estimate (3.49) holds.  $\square$

*Remark 3.3.* Since above dyadic decomposition and multilinear type estimates are standard processes, we will not repeat the detail proofs of similar estimates again but give the stated estimate directly.

**3.7. Energy estimate for the modified energy.** The goal of this subsection is to finish the energy estimate part by proving estimate (3.64).

Let's first prove the following lemma, which consists of various bilinear estimate. These estimates can be used directly in the proof of Lemma 3.5 and the later bootstrap argument for  $\phi_0$ .

**Lemma 3.4.** *For bilinear operator  $Q \in \{Q_{0,\mu}, Q_{\mu,\nu}, \tilde{Q}_{0,0}, \tilde{Q}_{0,\mu}, \tilde{Q}_{\mu,\nu}\}$ ,  $\tilde{Q} \in \{\tilde{Q}_0^1, \tilde{Q}_{0,\mu}^1, \tilde{Q}_{\mu,\nu}^1\}$  and any two smooth functions  $h_1, h_2 \in H^1 \cap W^{1+}$ , we have*

$$\|Q(h_1, h_2)\|_{L^2} \lesssim \min\{\|h_1\|_{H^1} \|h_2\|_{W^{1+}}, \|h_2\|_{H^1} \|h_1\|_{W^{1+}}\}, \quad (3.54)$$

$$\|\tilde{Q}(h_1, h_2)\|_{L^2} \lesssim \min\{\|h_1\|_{L^2} \|h_2\|_{W^{1+}}, \|h_2\|_{L^2} \|h_1\|_{W^{1+}}\}, \quad (3.55)$$

$$\begin{aligned} \|\tilde{Q}(h_1, h_2)\|_{L^\infty} &\lesssim \min\{\|h_1\|_{W^{1+}} \|h_2\|_{W^{1+}}, \min\{\|h_1\|_{L^\infty} \|h_2\|_{W^{1+}} + \|h_1\|_{L^\infty}^{3/4} \times \\ &\|h_1\|_{H^5}^{1/4} \|h_2\|_{L^\infty}^{3/4} \|h_2\|_{L^2}^{1/4}, \|h_2\|_{L^\infty} \|h_1\|_{W^{1+}} + \|h_2\|_{L^\infty}^{3/4} \|h_2\|_{H^5}^{1/4} \|h_1\|_{L^\infty}^{3/4} \|h_1\|_{L^2}^{1/4}\}\}. \end{aligned} \quad (3.56)$$

*Proof.* For any  $Q \in \{Q_{0,\mu}, Q_{\mu,\nu}, \tilde{Q}_{0,0}, \tilde{Q}_{0,\mu}, \tilde{Q}_{\mu,\nu}\}$ , from Lemma 2.2, we know that the associated symbol  $q(\xi - \eta, \eta)$  satisfies the following uniform bound,

$$\|q(\xi - \eta, \eta)\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{k_1+k_2}.$$

Similar to the proof of estimate (3.50), we have

$$\|Q(h_1, h_2)\|_{L^2} \lesssim \|h_1\|_{H^1} \|h_2\|_{W^{1+}},$$

and then we can switch the roles of  $h_1$  and  $h_2$  in above estimate to see (3.54) holds.

For any  $\tilde{Q} \in \{\tilde{Q}_0^1, \tilde{Q}_{0,\mu}^1, \tilde{Q}_{\mu,\nu}^1\}$ , from Lemma 2.2 we can see that the associated symbol  $\tilde{q}(\xi - \eta, \eta)$  satisfies the following uniform bound

$$\|\tilde{q}(\xi - \eta, \eta)\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{\min\{k_1, k_2\}},$$

with minor modifications, one can see the desired estimate (3.55) holds. Following the same idea, we have

$$\|\tilde{Q}(h_1, h_2)\|_{L^\infty} = \sum_{|k_1-k_2| \leq 4} 2^{k_1} 2^{-(1+)(k_1,+)} \|P_{k_1} h_1\|_{L^\infty} \|P_{k_2} h_2\|_{L^\infty} + \quad (3.57)$$

$$\sum_{|k_1-k_2| \geq 4} 2^{\min\{k_1, k_2\}} 2^{-(1+)(k_1,++k_2,+)} \|P_{k_1} h_1\|_{W^{1+}} \|P_{k_2} h_2\|_{W^{1+}} \lesssim \|h_1\|_{W^{1+}} \|h_2\|_{W^{1+}}. \quad (3.58)$$

There is another way to estimate the  $L^\infty$ -norm of  $\tilde{Q}(h_1, h_2)$ , which is as follows,

$$\|\tilde{Q}(h_1, h_2)\|_{L^\infty} \lesssim \|h_1\|_{L^\infty} \|h_2\|_{W^{1+}} + \quad (3.59)$$

$$\sum_{k_2 \leq k_1-4} 2^{k_2} 2^{-5k_1,+/4} (\|P_{k_1} h_1\|_{L^\infty} \|P_{k_2} h_2\|_{L^\infty})^{3/4} \|P_{k_2} h_2\|_{L^2}^{1/4} \|P_{k_1} h_1\|_{H^5}^{1/4} \quad (3.60)$$

$$\lesssim \|h_1\|_{L^\infty} \|h_2\|_{W^{1+}} + \|h_1\|_{L^\infty}^{3/4} \|h_1\|_{H^5}^{1/4} \|h_2\|_{L^\infty}^{3/4} \|h_2\|_{L^2}^{1/4}. \quad (3.61)$$

Since the upper bound we used is symmetric, we can switch the role of  $h_1$  and  $h_2$  in above estimates to see estimate (3.56) holds.  $\square$

From the definition of modified energy (3.48) and the construction of cubic correction terms, we have the following identity,

$$\begin{aligned} \frac{d}{dt} E^{modi}(t) &= \text{QAHigher}_1 + \text{QAHigher}_2, \quad \text{QAHigher}_1 = \frac{d}{dt} E(t) - [\text{“cubic” terms of } \frac{d}{dt} E(t)], \\ \text{QAHigher}_2 &= [\text{quartic and higher of } \frac{d}{dt} (E_{FCorr}(t) + E_{SCorr}(t))]. \end{aligned}$$

**Lemma 3.5.** *Under the bootstrap assumption (2.2), we have*

$$\sup_{t \in [0, T]} (1+t)^{-2p_0+1} |\text{QAHigher}_1| \lesssim \epsilon_0^2. \quad (3.62)$$

*Proof.* From (3.16), (3.19), (3.35), (3.36) and (3.42), we know that  $\text{QAHigher}_1$  is quadratic and higher in the sense of decay, i.e., we at least have one  $\phi_0$  among all inputs. Although there are many terms inside  $\text{QAHigher}_1$ , we can use the estimate (3.45) and the estimates in Lemma 3.4 to estimate all of them. As representative examples, we only choose two of them as follows.

- (i) For any tuples  $\beta, \gamma$ , such that  $|\beta|, |\gamma| < |\alpha| \in \{N_0, N_1 + 1\}$ , we have the following estimate by estimate (3.54) in Lemma 3.4, we have

$$\begin{aligned} \left| \int \overline{\Phi^\alpha}(t) \tilde{Q}_{0,\mu}(\phi_0^\beta(t), \Phi_\mu^\gamma(t)) dx \right| &\lesssim \|\Phi(t)\|_{X_{N_0}} [\|\phi_0(t)\|_{X_{N_0}} \|\Phi(t)\|_{Z'} + \|\Phi(t)\|_{X_{N_0}} \|\phi_0(t)\|_{Z'_1}] \\ &\lesssim (1+t)^{2p_0-1} \epsilon_1^3 \lesssim (1+t)^{2p_0-1} \epsilon_0^2. \end{aligned}$$

- (ii) From the  $\mathcal{S}_{k,k_1,k_2}^\infty$  norm estimate of  $q_2(\cdot, \cdot)$  in (3.34), we have

$$\begin{aligned} \left| \int \overline{\Phi^\alpha}(t) Q_2(R_2 \phi_0^\alpha(t), R_1 \Phi(t)) dx \right| &\lesssim \|\Phi(t)\|_{X_{N_0}} [\|\phi_0(t)\|_{X_{N_0}} \|\Phi(t)\|_{Z'} + \|\Phi(t)\|_{X_{N_0}} \|\phi_0(t)\|_{Z'_1}] \\ &\lesssim (1+t)^{2p_0-1} \epsilon_1^3 \lesssim (1+t)^{2p_0-1} \epsilon_0^2. \end{aligned}$$

□

**Lemma 3.6.** *Under the bootstrap assumption (2.2), we have*

$$\sup_{t \in [0, T]} (1+t)^{-2p_0+1} |\text{QAHigher}_2| \lesssim \epsilon_0^2. \quad (3.63)$$

Therefore, after combining this result with the results of Lemma 3.2 and Lemma 3.5, we have

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|(\phi_0, \Phi)\|_{X_{N_0}} \lesssim \epsilon_0. \quad (3.64)$$

**3.8. Proof of Lemma 3.6.** Note that  $\text{QAHigher}_2$  is quartic and higher, that is to say, the decay rate is sufficient and we only have to avoid losing derivatives to close the argument. After carefully checking  $E_{FCorr}(t)$  and  $E_{SCorr}(t)$ , we can see that only the following terms potentially lose derivatives,

$$\sum_{\substack{(|\alpha_1|+|\alpha_2|, |\alpha_3|+|\alpha_4|) \\ \in \{(N_0, 0), (N_1, 1)\}}} \sum_{\substack{\beta+\gamma=\alpha, |\gamma|=1, \\ |\gamma_3|=|\gamma_4|=0}} \binom{\alpha}{\beta} (J_{\beta,\gamma}^1(\Phi^\gamma) + J_{\beta,\gamma}^2(\Phi^\gamma) + J_{\beta,\gamma}^3(\Phi^\gamma)) + \frac{1}{4} (J_\alpha^1 + J_\alpha^2 + J_\alpha^3).$$

where

$$J_{\beta,\gamma}^1(\Phi^\gamma) := \text{Re} \left( \int \overline{\Gamma^\gamma \mathcal{N}_1^{(\beta,0)}} (A_{+,+}(\Phi^\beta, \Phi^\gamma) + A_{+,+}(\Phi^\gamma, \Phi^\beta) + A_{+,-}(\Phi^\beta, \overline{\Phi^\gamma})) \right)$$

$$\begin{aligned}
& + \overline{\Gamma^\gamma \mathcal{N}_0^{(\beta,0)}} (A_{+,+}(\phi_0^\beta, \Phi^\gamma) + A_{+,+}(\Phi^\gamma, \phi_0^\beta) + A_{+,-}(\phi_0^\beta, \overline{\Phi^\gamma})), \\
J_{\beta,\gamma}^2(\Phi^\gamma) &:= \operatorname{Re} \left( \int \overline{\Gamma^\gamma \Phi^\beta} (A_{+,+}(\mathcal{N}_1^{(\beta,0)}, \Phi^\gamma) + A_{+,+}(\Phi^\gamma, \mathcal{N}_1^{(\beta,0)}) + A_{+,-}(\mathcal{N}_1^{(\beta,0)}, \overline{\Phi^\gamma})) \right. \\
& \quad \left. + \overline{\Gamma^\gamma \phi_0^\beta} (A_{+,+}(\mathcal{N}_0^{(\beta,0)}, \Phi^\gamma) + A_{+,+}(\Phi^\gamma, \mathcal{N}_0^{(\beta,0)}) + A_{+,-}(\mathcal{N}_0^{(\beta,0)}, \overline{\Phi^\gamma})) \right), \\
J_{\beta,\gamma}^3(\Phi^\gamma) &:= \operatorname{Re} \left( \int \overline{\Gamma^\gamma \mathcal{N}_1^{(\beta,0)}} (A_{-,-}(\overline{\Phi^\beta}, \overline{\Phi^\gamma}) + A_{-,-}(\overline{\Phi^\gamma}, \overline{\Phi^\beta}) + A_{+,-}(\Phi^\gamma, \overline{\Phi^\beta})) \right. \\
& \quad \left. + \overline{\Gamma^\gamma \Phi^\beta} (A_{-,-}(\overline{\mathcal{N}_1^{(\beta,0)}}), \overline{\Phi^\gamma}) + A_{-,-}(\overline{\Phi^\gamma}, \overline{\mathcal{N}_1^{(\beta,0)}}) + A_{+,-}(\Phi^\gamma, \overline{\mathcal{N}_1^{(\beta,0)}})) \right), \\
J_\alpha^1 &:= \operatorname{Re} \left( \int \overline{\mathcal{N}_1^{(\alpha,0)}} (A_{+,+}^4(\Phi^\alpha, \Phi) + A_{+,-}^4(\Phi^\alpha, \overline{\Phi})) + \overline{\mathcal{N}_0^{(\alpha,0)}} (A_{+,+}^4(\phi_0^\alpha, \Phi) + A_{+,-}^4(\phi_0^\alpha, \overline{\Phi})) \right), \\
J_\alpha^2 &:= \operatorname{Re} \left( \int \overline{\Phi^\alpha} (A_{+,+}^4(\mathcal{N}_1^{(\alpha,0)}, \Phi) + A_{+,-}^4(\mathcal{N}_1^{(\alpha,0)}, \overline{\Phi})) + \overline{\phi_0^\alpha} (A_{+,+}^4(\mathcal{N}_0^{(\alpha,0)}, \Phi) + A_{+,-}^4(\mathcal{N}_0^{(\alpha,0)}, \overline{\Phi})) \right), \\
J_\alpha^3 &:= \operatorname{Re} \left( \int \overline{\mathcal{N}_1^{(\alpha,0)}} (A_{-,-}^4(\overline{\Phi^\alpha}, \overline{\Phi}) + A_{-,-}^4(\overline{\Phi}, \Phi)) + \overline{\Phi^\alpha} (A_{-,-}^4(\overline{\mathcal{N}_1^{(\alpha,0)}}), \overline{\Phi}) + A_{-,-}^4(\overline{\mathcal{N}_1^{(\alpha,0)}}), \Phi) \right).
\end{aligned}$$

The proof of Lemma 3.6 is splitted into three parts: (i) estimating  $J_{\beta,\gamma}^3(\Phi^\gamma)$  and  $J_\alpha^3$ , (ii) estimating  $J_{\beta,\gamma}^1(\Phi^\gamma)$  and  $J_{\beta,\gamma}^2(\Phi^\gamma)$ , (iii) estimating  $J_\alpha^1$  and  $J_\alpha^2$ . We address these three parts in the following three subsections respectively.

**3.8.1. Estimating  $J_{\beta,\gamma}^3(\Phi^\gamma)$  and  $J_\alpha^3$ .** We estimate these two terms as the starting point because they are relatively easier. A key observation for these terms is that we have better estimates for bilinear operators in  $J_{\beta,\gamma}^3(\Phi^\gamma)$  and  $J_\alpha^3$  in the worst scenario. More precisely, if  $|\eta| \ll |\xi| \sim |\xi - \eta|$ , we have

$$|\xi| + |\xi - \eta| \pm |\eta| \sim |\xi|, \quad \left\| \frac{1}{|\xi| + |\xi - \eta| \pm |\eta|} \right\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{-k_1}, \quad \text{if } k_2 \leq k_1 - 10,$$

which further gives us the following improved estimate

$$\|a_{-,-}(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{k_2}, \quad \|a_{-,-}^4(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{2k_2 - k_1}, \quad \text{if } k_2 \leq k_1 - 10. \quad (3.65)$$

If we switch the role of  $\eta$  and  $\xi - \eta$  in above argument, very similarly, we can derive the following estimate,

$$\|a_{+,-}(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{k_1}, \quad \|a_{+,-}^4(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{2k_1 - k_2}, \quad \text{if } k_1 \leq k_2 - 10. \quad (3.66)$$

With above better estimates: (3.65) and (3.66), we can show that  $J_\alpha^3$  and  $J_{\beta,\gamma}^3(\Phi^\gamma)$  actually do not lose derivative. More precisely, from (3.65), (3.66), and Lemma 2.1, we have

$$\begin{aligned}
|J_{\beta,\gamma}^3| + |J_\alpha^3| &\lesssim (\|\mathcal{N}_1^{(\beta,0)}\|_{L^2} + \|\mathcal{N}_1^{(\alpha,0)}\|_{H^{-1}}) \|\Phi\|_{X_{N_0}} \|\Phi\|_{Z'} \\
&\lesssim \|(\phi_0, \Phi)\|_{X_{N_0}} \|\Phi\|_{X_{N_0}} (\|\phi_0\|_{Z'_1} + \|\Phi\|_{Z'})^2 \lesssim (1+t)^{-1+2p_0} \epsilon_1^4 \lesssim (1+t)^{-1+2p_0} \epsilon_0^3.
\end{aligned}$$



3.8.2. *Estimating  $J_{\beta,\gamma}^1(\Phi^\gamma)$  and  $J_{\beta,\gamma}^2(\Phi^\gamma)$ .* Since  $|\gamma| = 1$ ,  $\Gamma^\gamma \in \{\partial_1, \partial_2\}$ , we write  $\Gamma^\gamma$  as  $\partial_j$  for some  $j \in \{1, 2\}$ . We first consider  $J_{\beta,\gamma}^1(\Phi^\gamma)$  and decompose it into two parts by splitting the input  $\partial_j \Phi$  into imaginary part and real part. More precisely, we have

$$J_{\beta,\gamma}^1(\Phi^\gamma) = J_{\beta,\gamma}^1(i\partial_j \text{Im}(\Phi)) + J_{\beta,\gamma}^1(\partial_j \text{Re}(\Phi)). \quad (3.67)$$

From the explicit formula of  $a_{\mu,\nu}(\cdot, \cdot)$  in (3.14), we can see the following facts,

$$a_{\mu,\nu}(\xi - \eta, \eta) = a_{\mu,\nu}(\eta - \xi, -\eta), \quad \text{Re}(a_{\mu,\nu}(\xi - \eta, \eta)) = 0, \quad (\mu, \nu) \in \mathcal{S}.$$

With these two facts, we know the following identity holds,

$$\text{Re}\left(\int f A_{\mu,\nu}(g, h)\right) = 0, \quad \text{if } f, g, \text{ and } h \text{ are all real.} \quad (3.68)$$

- *Estimate of  $J_{\beta,\gamma}^1(i\partial_j \text{Im}(\Phi))$ .* Using (3.68), we have the following identity for  $J_{\beta,\gamma}^1(i\partial_j \text{Im}(\Phi))$ ,

$$\begin{aligned} J_{\beta,\gamma}^1(i\partial_j \text{Im}(\Phi)) &= \text{Re}\left(\int \partial_j \text{Re}(\mathcal{N}_1^{(\beta,0)})(A_{+,+}(\text{Re}(\Phi^\beta), i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), \text{Re}(\Phi^\beta)) \right. \\ &\quad + A_{+,-}(\text{Re}(\Phi^\beta), -i\partial_j \text{Im}(\Phi))) + \partial_j \text{Re}(\mathcal{N}_0^{(\beta,0)})(A_{+,+}(\text{Re}(\phi_0^\beta), i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), \\ &\quad \text{Re}(\phi_0^\beta)) + A_{+,-}(\text{Re}(\phi_0^\beta), -i\partial_j \text{Im}(\Phi))) + \partial_j \text{Im}(\mathcal{N}_1^{(\beta,0)})(A_{+,+}(\text{Im}(\Phi^\beta), i\partial_j \text{Im}(\Phi)) \\ &\quad + A_{+,+}(i\partial_j \text{Im}(\Phi), \text{Im}(\Phi^\beta)) + A_{+,-}(\text{Im}(\Phi^\beta), -i\partial_j \text{Im}(\Phi))) + \partial_j \text{Im}(\mathcal{N}_0^{(\beta,0)})(A_{+,+}(\text{Im}(\phi_0^\beta), \\ &\quad i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), \text{Im}(\phi_0^\beta)) + A_{+,-}(\text{Im}(\phi_0^\beta), -i\partial_j \text{Im}(\Phi)))\Big). \end{aligned} \quad (3.69)$$

We will show that the cancellation that happened in (3.29), (3.31), and (3.34) also happens for  $J_{\beta,\gamma}^1(\partial_j \text{Im}(\Phi))$ . Intuitively speaking, we previously encountered “ $f^\beta \partial_j f^\beta h$ ” type terms, now we are dealing with “ $f^\beta \partial_j f^\beta h_1 h_2$ ” type terms. The key idea is still identifying symmetries.

From (1.15) and (1.19), we rewrite (3.69) in terms of  $\psi$ ,  $G_1$ , and  $G_2$  and have the following identity,

$$J_{\beta,\gamma}^1(i\partial_j \text{Im}(\Phi)) = J_{\beta,\gamma}^{1,1}(i\partial_j \text{Im}(\Phi)) + J_{\beta,\gamma}^{1,2}(i\partial_j \text{Im}(\Phi)), \quad (3.70)$$

where

$$\begin{aligned} J_{\beta,\gamma}^{1,1}(i\partial_j \text{Im}(\Phi)) &:= \text{Re}\left(\int \partial_j R_1 \tilde{\mathcal{N}}_1^{(\beta,0)}(A_{+,+}(R_2 G_2^\beta, i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), R_2 G_2^\beta) \right. \\ &\quad + A_{+,+}(R_2 G_2^\beta, -i\partial_j \text{Im}(\Phi))) - \partial_j R_2 \tilde{\mathcal{N}}_1^{(\beta,0)}(A_{+,+}(R_1 G_2^\beta, i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), R_1 G_2^\beta) \\ &\quad + A_{+,-}(R_1 G_2^\beta, -i\partial_j \text{Im}(\Phi))) + \partial_j R_2 \tilde{\mathcal{N}}_2^{(\beta,0)}(A_{+,+}(R_1 G_1^\beta, i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), R_1 G_1^\beta) \\ &\quad + A_{+,-}(R_1 G_1^\beta, -i\partial_j \text{Im}(\Phi))) - \partial_j R_1 \tilde{\mathcal{N}}_2^{(\beta,0)}(A_{+,+}(R_2 G_1^\beta, i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), R_2 G_1^\beta) \\ &\quad \left. + A_{+,-}(R_2 G_1^\beta, -i\partial_j \text{Im}(\Phi)))\right), \\ J_{\beta,\gamma}^{1,2}(i\partial_j \text{Im}(\Phi)) &:= \text{Re}\left(\int \partial_j \tilde{\mathcal{N}}_0^{(\beta,0)}(A_{+,+}(\psi^\beta, i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), \psi^\beta) \right. \\ &\quad + A_{+,-}(\psi^\beta, -i\partial_j \text{Im}(\Phi))) + \sum_{i=1,2} \partial_j R_i \tilde{\mathcal{N}}_1^{(\beta,0)}(A_{+,+}(R_i G_1^\beta, i\partial_j \text{Im}(\Phi)) + A_{+,+}(i\partial_j \text{Im}(\Phi), R_i G_1^\beta) \\ &\quad + A_{+,-}(R_i G_1^\beta, -i\partial_j \text{Im}(\Phi))) + \partial_j R_i \tilde{\mathcal{N}}_2^{(\beta,0)}(A_{+,+}(R_i G_2^\beta, i\partial_j \text{Im}(\Phi)) \\ &\quad \left. + A_{+,+}(i\partial_j \text{Im}(\Phi), R_i G_2^\beta) + A_{+,-}(R_i G_2^\beta, -i\partial_j \text{Im}(\Phi)))\right). \end{aligned}$$

We write  $J_{\beta,\gamma}^{1,1}(\partial_j \text{Im}(\Phi))$  on the Fourier side and have the following,

$$\begin{aligned} |J_{\beta,\gamma}^{1,1}(i\partial_j \text{Im}(\Phi))| &\lesssim \left| \int \int \overline{\widehat{\mathcal{N}_1^{(\beta,0)}}(\xi)} \widehat{G_2^\beta}(\xi - \eta) \widehat{\text{Im}(\Phi)}(\eta) m_1(\xi - \eta, \eta) d\eta d\xi \right| \\ &+ \left| \int \int \overline{\widehat{\mathcal{N}_2^{(\beta,0)}}(\xi)} \widehat{G_1^\beta}(\xi - \eta) \widehat{\text{Im}(\Phi)}(\eta) m_1(\xi - \eta, \eta) d\eta d\xi \right|, \end{aligned}$$

where

$$\begin{aligned} m_1(\xi - \eta, \eta) &= i\xi_j \eta_j \frac{\xi}{|\xi|} \times \left( \frac{\xi - \eta}{|\xi - \eta|} \right) (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) - a_{+,-}(\xi - \eta, \eta)) \\ &= i\xi_j \eta_j \frac{-\xi \times \eta}{|\xi||\xi - \eta|} (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) - a_{+,-}(\xi - \eta, \eta)). \end{aligned}$$

From above computation, we can see the cancellation comes from the Riesz operators. We have the following estimate from Lemma 2.2,

$$\|m_1(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{2k_2+k_1}, \quad \text{if } k_2 \leq k_1 - 10.$$

From above estimate and multilinear estimates in Lemma 2.1, we have the following estimate,

$$\begin{aligned} |J_{\beta,\gamma}^{1,1}(i\partial_j \text{Im}(\Phi))| &\lesssim \|(\tilde{\mathcal{N}}_1^{(\beta,0)}, \tilde{\mathcal{N}}_2^{(\beta,0)})\|_{L^2} (\|\Phi\|_{Z'} \|(G_1, G_2)\|_{X_{N_0}} + \|\Phi\|_{X_{N_0}} \|(G_1, G_2)\|_{Z'_1}) \\ &\lesssim \|(\phi_0, \Phi)\|_{X_{N_0}}^2 (\|\phi_0\|_{Z'_1} + \|\Phi\|_{Z'})^2 \lesssim (1+t)^{-2p_0+1} \epsilon_0^3. \end{aligned} \quad (3.71)$$

Now, we consider  $J_{\beta,\gamma}^{1,2}(\partial_j \text{Im}(\Phi))$ . Recall (3.30), similar to computations in (3.31), we can write  $J_{\beta,\gamma}^{1,2}(\partial_j \text{Im}(\Phi))$  on the Fourier side and have the following,

$$\begin{aligned} |J_{\beta,\gamma}^{1,2}(i\partial_j \text{Im}(\Phi))| &\lesssim \left| \int \int \int \overline{\widehat{\psi^\beta}(\xi - \sigma) \psi(\sigma) \widehat{\psi^\beta}(\xi - \eta)} \widehat{\text{Im}(\Phi)}(\eta) m_2(\xi, \eta, \sigma) d\sigma d\eta d\xi \right| \\ &+ \sum_{i=1,2} \left| \int \int \int \overline{\widehat{G_i^\beta}(\xi - \sigma) \widehat{\psi}(\sigma) \widehat{G_i^\beta}(\xi - \eta)} \widehat{\text{Im}(\Phi)}(\eta) m_3(\xi, \eta, \sigma) d\sigma d\eta d\xi \right| \\ &+ \left| \int \int \int \overline{\widehat{G_i^\beta}(\xi - \sigma) \widehat{G_i}(\sigma) \widehat{\psi^\beta}(\xi - \eta)} \widehat{\text{Im}(\Phi)}(\eta) m_4(\xi, \eta, \sigma) d\sigma d\eta d\xi \right|, \end{aligned} \quad (3.72)$$

where

$$\begin{aligned} m_2(\xi, \eta, \sigma) &= i\xi_j \eta_j (q(\xi - \sigma, \sigma) + q(\sigma, \xi - \sigma)) (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta)) \\ &+ i(\xi_j - \eta_j - \sigma_j)(-\eta_j) (q(\xi - \eta, -\sigma) + q(-\sigma, \xi - \eta)) \overline{a_{+,-}(\xi - \sigma, -\eta)}, \\ m_3(\xi, \eta, \sigma) &= i\xi_j \eta_j \frac{(\xi - \sigma) \times \sigma}{2\pi} \frac{\xi \cdot (\xi - \eta)}{|\xi||\xi - \eta|} (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta)) \\ &+ i(\xi_j - \eta_j - \sigma_j)(-\eta_j) \frac{(\xi - \eta) \times (-\sigma)}{2\pi} \frac{(\xi - \eta - \sigma) \cdot (\xi - \sigma)}{|\xi - \eta - \sigma||\xi - \sigma|} \overline{a_{+,-}(\xi - \sigma, -\eta)}, \end{aligned} \quad (3.73)$$

$$\begin{aligned} m_4(\xi, \eta, \sigma) &= i\xi_j \eta_j (-q(\xi - \sigma, \sigma) - q(\sigma, \xi - \sigma)) \frac{\xi \cdot (\xi - \eta)}{|\xi||\xi - \eta|} (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) - a_{+,-}(\xi - \eta, \eta)) \\ &+ i(\xi_j - \eta_j - \sigma_j)(-\eta_j) \frac{(-\sigma) \times (\xi - \eta)}{2\pi} \frac{(\xi - \eta - \sigma) \cdot (\xi - \eta)}{|\xi - \eta - \sigma||\xi - \eta|} \overline{(a_{+,-}(\xi - \sigma, -\eta) - a_{+,+}(\xi - \sigma, -\eta) - a_{+,+}(-\eta, \xi - \sigma))} \end{aligned} \quad (3.74)$$

For (3.72), losing derivative is only relevant when  $|\eta|, |\sigma| \ll |\xi|$ , hence we assume  $|\eta|, |\sigma| \ll |\xi|$  in the rest of this subsection.

To see the cancellation structures inside above symbols, we first point out the following key cancellation. From (3.9) and (3.10), we have

$$\begin{aligned} & \tilde{m}'_{+,+}(\xi - \eta, \eta) + \tilde{m}'_{+,+}(\eta, \xi - \eta) + \tilde{m}'_{+,-}(\xi, -\eta) \\ &= \frac{-\xi \cdot (\xi - 2\eta)}{8\pi|\xi|^2} (1 - \cos(\angle(\xi - \eta, \eta))) ((\xi - \eta) \times \eta) - \frac{\xi}{8\pi|\xi|} \cdot \left( \frac{\xi - \eta}{|\xi - \eta|} - \frac{\eta}{|\eta|} \right) ((\xi - \eta) \times \eta) \\ &+ \frac{-(\xi - \eta) \cdot (\xi - \eta + 2\eta)}{8\pi|\xi - \eta|^2} (1 + \cos(\angle(\xi, -\eta))) (\xi \times (-\eta)) - \frac{\xi - \eta}{8\pi|\xi - \eta|} \cdot \left( \frac{\xi}{|\xi|} + \frac{\eta}{|\eta|} \right) (\xi \times (-\eta)) \\ &= \left[ \frac{\xi \cdot \eta}{4\pi|\xi|^2} (1 - \cos(\angle(\xi - \eta, \eta))) - \frac{(\xi - \eta) \cdot \eta}{4\pi|\xi - \eta|^2} (1 + \cos(\angle(\xi, -\eta))) \right] ((\xi - \eta) \times \eta), \end{aligned}$$

therefore,

$$\begin{aligned} a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) + \overline{a_{+,-}(\xi, -\eta)} &= \frac{i(\tilde{m}'_{+,+}(\xi - \eta, \eta) + \tilde{m}'_{+,+}(\eta, \xi - \eta) + \tilde{m}'_{+,-}(\xi, -\eta))}{|\xi| - |\xi - \eta| - |\eta|} \\ &= i \left[ -\frac{(|\xi| + |\xi - \eta| + |\eta|)\xi \cdot \eta}{8\pi|\xi - \eta||\eta||\xi|^2} + \frac{(|\xi| + |\xi - \eta| - |\eta|)((\xi - \eta) \cdot \eta)}{8\pi|\xi||\eta||\xi - \eta|^2} \right] ((\xi - \eta) \times \eta). \end{aligned} \quad (3.75)$$

From above explicit formula and Lemma 2.2, we have the following estimate,

$$\|a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) + \overline{a_{+,-}(\xi, -\eta)}\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{k_2}, \quad \text{if } k_2 \leq k_1 - 10. \quad (3.76)$$

Recall that  $\|a_{\mu,\nu}(\xi - \eta, \eta)\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{\max\{k_1, k_2\}}$ , comparing it with the first estimate of (3.76), we know that the cancellation indeed happens.

Note that

$$\begin{aligned} & a_{+,+}(\xi, -\eta) + a_{+,+}(\xi, -\eta) + \overline{a_{+,-}(\xi - \eta, \eta)} \\ &= a_{+,+}(\xi - \eta - (-\eta), -\eta) + a_{+,+}(\xi - \eta - (-\eta), -\eta) + \overline{a_{+,-}(\xi - \eta, -(-\eta))}, \end{aligned} \quad (3.77)$$

after changing of coordinates in (3.75), we have the following byproduct,

$$\|a_{+,+}(\xi, -\eta) + a_{+,+}(\xi, -\eta) + \overline{a_{+,-}(\xi - \eta, \eta)}\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{k_2}, \quad \text{if } k_2 \leq k_1 - 10. \quad (3.78)$$

To see cancellation inside  $m_i(\xi, \eta, \sigma), i \in \{2, 3, 4\}$  we rewrite them (see (3.73) and (3.74)) as follows,

$$\begin{aligned} m_2(\xi, \eta, \sigma) &= i\xi_j \eta_j \underbrace{(q(\xi - \sigma, \sigma) + q(\sigma, \xi - \sigma) + q(\xi, -\sigma) + q(-\sigma, \xi))}_{\text{cancellation from (3.33)}} (a_{+,+}(\xi - \eta, \eta) \\ &+ a_{+,+}(\eta, \xi - \eta)) - i\xi_j \eta_j (q(\xi, -\sigma) + q(-\sigma, \xi)) \underbrace{(a_{+,+}(\xi - \eta, \eta) + a_{+,-}(\eta, \xi - \eta) + \overline{a_{+,-}(\xi, -\eta)})}_{\text{cancellation from (3.75)}} \\ &+ \underbrace{i(\eta_j + \sigma_j) \eta_j (q(\xi - \eta, -\sigma) + q(-\sigma, \xi - \eta)) \overline{a_{+,-}(\xi - \sigma, -\eta)}}_{\text{rough estimate will do}} - i\xi_j \eta_j \overline{a_{+,-}(\xi, -\eta)} \times \\ &\quad \underbrace{(q(\xi - \eta, -\sigma) + q(-\sigma, \xi - \eta) - q(\xi, -\sigma) - q(-\sigma, \xi))}_{\text{cancellation from the smallness of the difference between } \xi - \eta \text{ and } \xi} + i\xi_j \eta_j (q(\xi - \eta, -\sigma) \\ &\quad + q(-\sigma, \xi - \eta)) \underbrace{(\overline{a_{+,-}(\xi, -\eta)} - \overline{a_{+,-}(\xi - \sigma, -\eta)})}_{\text{cancellation from the smallness of the difference between } \xi - \sigma \text{ and } \xi}. \end{aligned} \quad (3.79)$$

From above decomposition, estimates (3.34) and (3.76), Lemma 2.2, we have

$$\|m_2(\xi, \eta, \sigma)\|_{\mathcal{S}_{k, k_1, k_2, k_3}^\infty} \lesssim 2^{3 \max\{k_2, k_3\} + 2k_1}, \quad \text{if } \max\{k_2, k_3\} \leq k_1 - 10. \quad (3.80)$$

For simplicity, we only have to check these terms inside  $m_3(\xi, \eta, \sigma)$  and  $m_4(\xi, \eta, \sigma)$  that do not satisfy (3.80) type estimate and ignore those terms that already satisfy (3.80) type estimate. As a result, we have

$$m_3(\xi, \eta, \sigma) = i\xi_j \eta_j \frac{\xi \times \sigma}{2\pi} (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) + \overline{a_{+,-}(\xi, -\eta)} - \overline{a_{+,-}(\xi, -\eta)} + \overline{a_{+,-}(\xi - \sigma, -\eta)}) \\ + \text{other terms inside (3.73) that satisfy (3.80) type estimate,}$$

$$m_4(\xi, \eta, \sigma) = i\xi_j \eta_j \underbrace{\left[ \frac{(-\sigma) \times \xi}{2\pi} - q(\xi - \sigma, \sigma) - q(\sigma, \xi - \sigma) \right]}_{\text{cancellation from (3.32)}} (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) \\ - a_{+,-}(\xi - \eta, -\eta)) - i\xi_j \eta_j \frac{(-\sigma) \times \xi}{2\pi} \left( \underbrace{(a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta) + \overline{a_{+,-}(\xi, -\eta)})}_{\text{cancellation from (3.75)}} \right. \\ \left. - \underbrace{(\overline{a_{+,+}(\xi, -\eta)} + \overline{a_{+,+}(\xi, -\eta)} + a_{+,-}(\xi - \eta, \eta))}_{\text{cancellation from (3.78)}} + \underbrace{(-\overline{a_{+,-}(\xi, -\eta)} + \overline{a_{+,-}(\xi - \sigma, -\eta)})}_{\text{cancellation from the smallness of } |\sigma|} \right. \\ \left. - \underbrace{(\overline{a_{+,+}(\xi - \sigma, -\eta)} + \overline{a_{+,+}(\xi, -\eta)} - \overline{a_{+,+}(-\eta, \xi - \sigma)} + \overline{a_{+,+}(-\eta, \xi)})}_{\text{cancellation from the smallness of } |\sigma|} \right) \\ + \text{other terms inside (3.74) that satisfy (3.80) type estimate.}$$

From above decomposition, estimates (3.34) and (3.76), Lemma 2.2, we have

$$\|m_3(\xi, \eta, \sigma)\|_{\mathcal{S}_{k, k_1, k_2, k_3}^\infty} + \|m_4(\xi, \eta, \sigma)\|_{\mathcal{S}_{k, k_1, k_2, k_3}^\infty} \lesssim 2^{3 \max\{k_2, k_3\} + 2k_1}, \quad \text{if } \max\{k_2, k_3\} \leq k_1 - 10. \quad (3.81)$$

From estimates (3.80) and (3.81) on symbols and estimates in Lemma 2.1, we have

$$|J_{\beta, \gamma}^{1,2}(i\partial_j \text{Im}(\Phi))| \lesssim (\|\Phi\|_{Z'} \|(G_1, G_2)\|_{X_{N_0}} + \|\Phi\|_{X_{N_0}} \|(G_1, G_2)\|_{Z_1'})^2 \lesssim (1+t)^{-1+2p_0} \epsilon_0^3. \quad (3.82)$$

• *Estimate of  $J_{\beta, \gamma}^1(\partial_j \text{Re}(\Phi))$ .* Now, we proceed to estimate  $J_{\beta, \gamma}^1(\partial_j \text{Re}(\Phi))$ . From (3.68) and the fact that  $\text{Im}(\phi_0) = 0$ , we have the following identity,

$$J_{\beta, \gamma}^1(\partial_j \text{Re}(\Phi)) = \text{Re} \left( \int \partial_j \text{Re}(\mathcal{N}_1^{(\beta, 0)}) (A_{+,+}(i\text{Im}(\Phi^\beta), \partial_j \text{Re}(\Phi)) + A_{+,+}(\partial_j \text{Re}(\Phi), i\text{Im}(\Phi^\beta)) \right. \\ \left. + A_{+,-}(i\text{Im}(\Phi^\beta), \partial_j \text{Re}(\Phi)) \right) + \text{Re} \left( \int -i\partial_j \text{Im}(\mathcal{N}_1^{(\beta, 0)}) (A_{+,+}(\text{Re}(\Phi^\beta), \partial_j \text{Re}(\Phi)) \right. \\ \left. + A_{+,+}(\partial_j \text{Re}(\Phi), \text{Re}(\Phi^\beta)) + A_{+,-}(\text{Re}(\Phi^\beta), \partial_j \text{Re}(\Phi)) \right).$$

From (1.15) and (1.19), we represent  $J_{\beta, \gamma}^1(\partial_j \text{Re}(\Phi))$  in terms of  $\psi$ ,  $G_1$ , and  $G_2$  and then write it on the Fourier side. As a result, we have

$$|J_{\beta, \gamma}^1(\partial_j \text{Re}(\Phi))| \lesssim \sum_{i=1,2} \left| \int \int \int \widehat{\psi^\beta}(\xi - \sigma) \widehat{G_i}(\sigma) \widehat{\psi^\beta}(\xi - \eta) \widehat{\psi}(\eta) c_i^1(\xi, \eta, \sigma) d\sigma d\eta d\xi \right| +$$

$$\begin{aligned} & \sum_{i=1,2} \left| \int \int \int \overline{\widehat{\psi}^\beta(\xi - \sigma) \widehat{\psi}(\sigma) \widehat{G}_i^\beta(\xi - \eta) \widehat{\psi}(\eta)} c_i^2(\xi, \eta, \sigma) d\eta d\sigma d\xi \right| \\ & + \sum_{m,n=1,2} \left| \int \int \int \overline{\widehat{G}_m^\beta(\xi - \sigma) \widehat{G}_m(\sigma) \widehat{G}_n^\beta(\xi - \eta) \widehat{\psi}(\eta)} c_{m,n}^3(\xi, \eta, \sigma) d\sigma d\eta d\xi \right|, \end{aligned}$$

where

$$\begin{aligned} c_i^1(\xi, \eta, \sigma) &= \frac{-\xi_i \xi_j \eta_j}{|\xi|} \frac{(\sigma \times (\xi - \sigma))}{2\pi} (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta)) \\ &+ \frac{-(\xi_i - \eta_i - \sigma_i)(\xi_j - \eta_j - \sigma_j)(-\eta_j)}{|\xi - \eta - \sigma|} \frac{((- \sigma) \times (\xi - \sigma))}{2\pi} \overline{a_{+,-}(\xi - \sigma, -\eta)}, \\ c_i^2(\xi, \eta, \sigma) &= \frac{\xi_i \xi_j \eta_j}{|\xi|} (q(\xi - \sigma, \sigma) + q(\sigma, \xi - \sigma)) (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta)) \\ &+ \frac{(\xi_i - \eta_i - \sigma_i)(\xi_j - \eta_j - \sigma_j)(-\eta_j)}{|\xi - \eta - \sigma|} (q(\xi - \eta, -\sigma) + q(-\sigma, \xi - \eta)) \overline{a_{+,-}(\xi - \sigma, -\eta)} \\ &+ \frac{\xi_i \xi_j \sigma_j}{|\xi|} \frac{((\xi - \eta) \times \eta)}{2\pi} (a_{+,+}(\xi - \sigma, \sigma) + a_{+,+}(\sigma, \xi - \sigma)) \\ &+ \frac{(\xi_i - \sigma_i - \eta_i)(\xi_j - \eta_j - \sigma_j)(-\sigma_j)}{|\xi - \eta - \sigma|} \frac{((\xi - \sigma) \times (-\eta))}{2\pi} \overline{a_{+,-}(\xi - \eta, -\sigma)}, \\ c_{m,n}^3(\xi, \eta, \sigma) &= \frac{\xi_m(\xi_n - \eta_m)\xi_j\eta_j}{|\xi||\xi - \eta|} (-q(\xi - \sigma, \sigma) - q(\sigma, \xi - \sigma)) (a_{+,+}(\xi - \eta, \eta) + a_{+,+}(\eta, \xi - \eta)) \\ &+ \frac{(\xi_m - \eta_m - \sigma_m)(\xi_n - \sigma_n)(\xi_j - \eta_j - \sigma_j)(-\eta_j)}{|\xi - \eta - \sigma||\xi - \sigma|} (-q(\xi - \eta, -\sigma) - q(\xi - \eta, -\sigma)) \overline{a_{+,-}(\xi - \sigma, -\eta)}. \end{aligned}$$

Very similar to the decompositions we did for  $m_i(\xi, \eta, \sigma)$ ,  $i \in \{2, 3, 4\}$ , we can do similar decompositions for  $c_i^1(\xi, \eta, \sigma)$ ,  $c_i^2(\xi, \eta, \sigma)$ , and  $c_{m,n}^3(\xi, \eta, \sigma)$ ,  $i, m, n \in \{1, 2\}$ , to see cancellations inside, we omit details here. As a result, we have the following estimate,

$$\begin{aligned} & \sum_{i,m,n=1,2} \|c_i^1(\xi, \eta, \sigma)\|_{S_{k,k_1,k_2,k_3}^\infty} + \|c_i^2(\xi, \eta, \sigma)\|_{S_{k,k_1,k_2,k_3}^\infty} + \|c_{m,n}^3(\xi, \eta, \sigma)\|_{S_{k,k_1,k_2,k_3}^\infty} \\ & \lesssim 2^{3 \max\{k_2, k_3\} + 2k_1}, \quad \text{if } \max\{k_2, k_3\} \leq k_1 - 10. \end{aligned}$$

From above estimate and multilinear estimate in Lemma 2.1, we have

$$|J_{\beta,\gamma}^{1,2}(\partial_j \text{Re}(\Phi))| \lesssim (\|\Phi\|_{Z'} \|(G_1, G_2)\|_{X_{N_0}} + \|\Phi\|_{X_{N_0}} \|(G_1, G_2)\|_{Z_1'})^2 \lesssim (1+t)^{-1+2p_0} \epsilon_0^3. \quad (3.83)$$

To sum up, from (3.67), (3.70), (3.71), (3.82), and (3.83) we have

$$|J_{\beta,\gamma}^1(\Phi^\gamma)| \lesssim (1+t)^{-1+2p_0} \epsilon_0^3.$$

After noting the following two facts, one can estimate  $J_{\beta,\gamma}^2(\Phi^\gamma)$  very similarly with minor modifications.

(i) We can transform  $J_{\beta,\gamma}^2(\Phi^\gamma)$  into a term, which is similar to  $J_{\beta,\gamma}^1(\overline{\Phi}^\gamma)$  by the following equality,

$$\text{Re} \left( \int \overline{\Gamma^\gamma} f(T(g, \Gamma^\gamma h)) \right) = \text{Re} \left( \int \overline{\Gamma^\gamma} g \tilde{T}(f, \overline{\Gamma^\gamma h}) \right) + \text{a term that doesn't lose derivative},$$

where  $\tilde{T}(\cdot, \cdot)$  is defined by the following symbol

$$\tilde{t}(\xi - \eta, \eta) := \overline{t(\xi, -\eta)}, \quad t(\xi - \eta, \eta) \text{ is the symbol of bilinear operator } T \in \{A_{+,+}, A_{+,-}\}.$$

(ii) From (3.77) and (3.78), we can see that the cancellation also happens for the following term when  $|\eta| \ll |\xi|$ ,

$$\tilde{a}_{+,+}(\xi - \eta, \eta) + \tilde{a}_{+,+}(\eta, \xi - \eta) + \overline{\tilde{a}_{+,-}(\xi, -\eta)} = \overline{a_{+,+}(\xi, -\eta)} + \overline{a_{+,+}(-\eta, \xi)} + a_{+,-}(\xi - \eta, \eta).$$

3.8.3. *Estimating  $J_\alpha^1$  and  $J_\alpha^2$ .* We first estimate  $J_\alpha^1$  and decompose it into two parts as follows,

$$\begin{aligned} J_\alpha^1 &= J_\alpha^1(\operatorname{Re}(\Phi)) + J_\alpha^1(i\operatorname{Im}(\Phi)), \quad J_\alpha^1(\operatorname{Re}(\Phi)) := \operatorname{Re} \left( \int \overline{\mathcal{N}_1^{(\alpha,0)}} (A_{+,+}^4(\Phi^\alpha, \operatorname{Re}(\Phi)) \right. \\ &\quad \left. + A_{+,-}^4(\Phi^\alpha, \operatorname{Re}(\Phi)) + \overline{\mathcal{N}_0^{(\alpha,0)}} (A_{+,+}^4(\phi_0^\alpha, \operatorname{Re}(\Phi)) + A_{+,-}^4(\phi_0^\alpha, \operatorname{Re}(\Phi))) \right), \\ J_\alpha^1(i\operatorname{Im}(\Phi)) &:= \operatorname{Re} \left( \int \overline{\mathcal{N}_1^{(\alpha,0)}} (A_{+,+}^4(\Phi^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(\Phi^\alpha, -i\operatorname{Im}(\Phi)) \right. \\ &\quad \left. + \overline{\mathcal{N}_0^{(\alpha,0)}} (A_{+,+}^4(\phi_0^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(\phi_0^\alpha, -i\operatorname{Im}(\Phi))) \right). \end{aligned}$$

Recall the detail formula of the symbol of  $A_{\nu,\kappa}^4(\cdot, \cdot)$  in (3.44), we have the following identity

$$a_{+,+}^4(\xi - \eta, \eta) + a_{+,-}^4(\xi - \eta, \eta) = \frac{iq_2(\xi - \eta, \eta)}{|\xi - \eta|}.$$

From Lemma 2.2, we have

$$\|a_{+,+}^4(\xi - \eta, \eta) + a_{+,-}^4(\xi - \eta, \eta)\|_{\mathcal{S}_{k,k_1,k_2}^\infty} \lesssim 2^{2k_2-k_1}, \quad \text{if } k_2 \leq k_1 - 10,$$

which further gives us the following estimate,

$$|J_\alpha^1(\operatorname{Re}(\Phi))| \lesssim \|\mathcal{N}_1^{(\alpha,0)}\|_{H^{-1}} \|\Phi\|_{X_{N_0}} \|\Phi\|_{Z'} \lesssim \|(\phi_0, \Phi)\|_{X_{N_0}}^2 (\|\phi_0\|_{Z'_1} + \|\Phi\|_{Z'})^2 \lesssim (1 + |t|)^{-1+2p_0} \epsilon_0^3. \quad (3.84)$$

By using identity (3.68), we have

$$\begin{aligned} J_\alpha^1(i\operatorname{Im}(\Phi)) &= J_\alpha^{1,1}(i\operatorname{Im}(\Phi)) + J_\alpha^{1,2}(i\operatorname{Im}(\Phi)), \\ J_\alpha^{1,1}(i\operatorname{Im}(\Phi)) &:= \operatorname{Re} \left( \int R_2 \tilde{\mathcal{N}}_2^{(\alpha,0)} (A_{+,+}^4(R_1 G_1^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(R_1 G_1^\alpha, -i\operatorname{Im}(\Phi))) \right. \\ &\quad \left. - R_1 \tilde{\mathcal{N}}_2^{(\alpha,0)} (A_{+,+}^4(R_2 G_1^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(R_2 G_1^\alpha, -i\operatorname{Im}(\Phi))) + R_1 \tilde{\mathcal{N}}_1^{(\alpha,0)} (A_{+,+}^4(R_2 G_2^\alpha, i\operatorname{Im}(\Phi)) \right. \\ &\quad \left. + A_{+,-}^4(R_2 G_2^\alpha, -i\operatorname{Im}(\Phi))) - R_2 \tilde{\mathcal{N}}_1^{(\alpha,0)} (A_{+,+}^4(R_1 G_2^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(R_1 G_2^\alpha, -i\operatorname{Im}(\Phi))) \right), \\ J_\alpha^{1,2}(i\operatorname{Im}(\Phi)) &:= \operatorname{Re} \left( \int \tilde{\mathcal{N}}_0^{(\alpha,0)} (A_{+,+}^4(\psi^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(\psi^\alpha, -i\operatorname{Im}(\Phi))) \right. \\ &\quad \left. + \sum_{i,j=1,2} R_i \tilde{\mathcal{N}}_j^{(\alpha,0)} (A_{+,+}^4(R_i G_j^\alpha, i\operatorname{Im}(\Phi)) + A_{+,-}^4(R_i G_j^\alpha, -i\operatorname{Im}(\Phi))) \right). \end{aligned}$$

Very similar to estimate of  $J_{\beta,\gamma}^{1,1}(\partial_j \operatorname{Im}(\Phi))$  in (3.71), with minor modifications, we can prove the following estimate,

$$\begin{aligned} |J_\alpha^{1,1}(i\operatorname{Im}(\Phi))| &\lesssim \|(\tilde{\mathcal{N}}_1^{(\alpha,0)}, \tilde{\mathcal{N}}_2^{(\alpha,0)})\|_{H^{-1}} \|(G_1, G_2, \Phi)\|_{X_{N_0}} (\|(G_1, G_2)\|_{Z'_1} + \|\Phi\|_{Z'}) \\ &\lesssim \|(\phi_0, \Phi)\|_{X_{N_0}}^2 (\|\phi_0\|_{Z'_1} + \|\Phi\|_{Z'}) \lesssim (1 + t)^{-1+2p_0} \epsilon_0^3. \end{aligned} \quad (3.85)$$

Before proceeding to estimate  $J_\alpha^{1,2}(i\text{Im}(\Phi))$ , we first point out the following key cancellation when  $|\eta| \ll |\xi|$ ,

$$\begin{aligned} a_{+,+}^4(\xi - \eta, \eta) - \overline{a_{+,+}^4(\xi, -\eta)} &= \frac{i(|\xi| + |\xi - \eta| + |\eta|)}{2|\xi - \eta||\eta|} \left( \frac{\xi \cdot \eta}{\pi|\xi|^2}(\xi \times \eta) - \frac{(\xi - \eta) \cdot \eta}{\pi|\xi - \eta|^2}(\xi \times \eta) \right) - \\ &\quad - \left( \frac{i(|\xi| + |\xi - \eta| - |\eta|)}{2|\xi||\eta|} - \frac{i(|\xi| + |\xi - \eta| + |\eta|)}{2|\xi - \eta||\eta|} \right) \frac{(\xi - \eta) \cdot \eta}{\pi|\xi - \eta|^2}(\xi \times \eta), \end{aligned}$$

above computation follows directly from (3.32) and (3.44). From Lemma 2.2, we have

$$\|a_{+,+}^4(\xi - \eta, \eta) - \overline{a_{+,+}^4(\xi, -\eta)}\|_{S_{k,k_1,k_2}^\infty} \lesssim 2^{2k_2-k_1}, \quad \text{if } k_2 \leq k_1 - 10. \quad (3.86)$$

Very Similar to what we did for  $J_{\beta,\gamma}^{1,2}(i\partial_j \text{Im}(\Phi))$  in (3.72), we can write  $J_\alpha^{1,2}(i\text{Im}(\Phi))$  on the Fourier side and rewrite associated symbols as follows,

$$\begin{aligned} |J_\alpha^{1,2}(i\text{Im}(\Phi))| &\lesssim \left| \int \int \int \widehat{\psi^\beta}(\xi - \sigma) \widehat{\psi}(\sigma) \widehat{\psi^\beta}(\xi - \eta) \widehat{\text{Im}(\Phi)}(\eta) m_5(\xi, \eta, \sigma) d\sigma d\eta d\xi \right| \\ &\quad + \sum_{i=1,2} \left| \int \int \int \widehat{G_i^\beta}(\xi - \sigma) \widehat{\psi}(\sigma) \widehat{G_i^\beta}(\xi - \eta) \widehat{\text{Im}(\Phi)}(\eta) m_6(\xi, \eta, \sigma) d\sigma d\eta d\xi \right| \\ &\quad + \left| \int \int \int \widehat{G_i^\beta}(\xi - \sigma) \widehat{G_i}(\sigma) \widehat{\psi^\beta}(\xi - \eta) \widehat{\text{Im}(\Phi)}(\eta) m_7(\xi, \eta, \sigma) d\sigma d\eta d\xi \right|, \end{aligned}$$

where

$$\begin{aligned} m_5(\xi, \eta, \sigma) &= i(q(\xi - \sigma, \sigma) + q(\sigma, \xi - \sigma) + q(\xi, -\sigma) + q(-\sigma, \xi)) a_{+,+}^4(\xi - \eta, \eta) + i(q(\xi - \eta, -\sigma) + q(-\sigma, \xi - \eta) \\ &\quad - q(\xi, -\sigma) - q(-\sigma, \xi)) a_{+,+}^4(\xi - \eta, \eta) + i(q(\xi - \eta, -\sigma) + q(-\sigma, \xi - \eta)) (\overline{a_{+,-}^4(\xi - \sigma, -\eta)} - a_{+,+}^4(\xi - \eta, \eta)), \\ m_6(\xi, \eta, \sigma) &= i \frac{(\xi - \sigma) \times \sigma}{2\pi} \frac{\xi \cdot (\xi - \eta)}{|\xi||\xi - \eta|} a_{+,+}^4(\xi - \eta, \eta) \\ &\quad + i \frac{(\xi - \eta) \times (-\sigma)}{2\pi} \frac{(\xi - \eta - \sigma) \cdot (\xi - \sigma)}{|\xi - \eta - \sigma||\xi - \sigma|} \overline{a_{+,-}^4(\xi - \sigma, -\eta)} \\ &= i \frac{\xi \times \sigma}{2\pi} \left( a_{+,+}^4(\xi - \eta, \eta) - \overline{a_{+,-}^4(\xi, -\eta)} + \overline{a_{+,-}^4(\xi, -\eta)} - \overline{a_{+,-}^4(\xi - \sigma, -\eta)} \right) + \text{other terms}, \\ m_7(\xi, \eta, \sigma) &= (-q(\xi - \sigma, \sigma) - q(\sigma, \xi - \sigma)) \frac{\xi \cdot (\xi - \eta)}{|\xi||\xi - \eta|} a_{+,+}^4(\xi - \eta, \eta) + i \frac{(-\sigma) \times (\xi - \eta)}{2\pi} \times \\ &\quad \frac{(\xi - \eta - \sigma) \cdot (\xi - \eta)}{|\xi - \eta - \sigma||\xi - \eta|} \overline{a_{+,-}^4(\xi - \sigma, -\eta)} = \left( \frac{(-\sigma) \times \xi}{2\pi} - q(\xi - \sigma, \sigma) - q(\sigma, \xi - \sigma) \right) a_{+,+}^4(\xi - \eta, \eta) \\ &\quad + \frac{(-\sigma) \times \xi}{2\pi} \left( a_{+,+}^4(\xi - \eta, \eta) - \overline{a_{+,-}^4(\xi, -\eta)} + \overline{a_{+,-}^4(\xi, -\eta)} - \overline{a_{+,-}^4(\xi - \sigma, -\eta)} \right) + \text{other terms}. \end{aligned}$$

Since the structures inside are very similar to what we encountered in the  $J_{\beta,\gamma}^{1,2}(\partial_j \text{Im}(\Phi))$  case, we only highlighted the cancellations in symbols above and omitted the detail formulas for good error terms. From above decomposition, (3.32), (3.33), estimates (3.34), (3.45), and (3.86) and Lemma 2.2, we have the following estimate when  $\max\{k_2, k_3\} \leq k_1 - 10$ ,

$$\|m_5(\xi, \eta, \sigma)\|_{S_{k,k_1,k_2,k_3}^\infty} + \|m_6(\xi, \eta, \sigma)\|_{S_{k,k_1,k_2,k_3}^\infty} + \|m_7(\xi, \eta, \sigma)\|_{S_{k,k_1,k_2,k_3}^\infty} \lesssim 2^{3\max\{k_2,k_3\}},$$

which further gives us the following estimate,

$$\begin{aligned} |J_\alpha^{1,2}(i\text{Im}(\Phi))| &\lesssim \|(G_1, G_2, \psi, \Phi)\|_{X_{N_0}}^2 (\|(G_1, G_2, \psi)\|_{Z'_1} + \|\Phi\|_{Z'})^2 \\ &\lesssim \|(\phi_0, \Phi)\|_{X_{N_0}}^2 (\|\phi_0\|_{Z'_1} + \|\Phi\|_{Z'})^2 \lesssim (1+t)^{-1+2p_0} \epsilon_1^4 \lesssim (1+t)^{-1+2p_0} \epsilon_0^3. \end{aligned} \quad (3.87)$$

To sum up, from (3.84), (3.85), and (3.87), we have

$$|J_\alpha^1| \lesssim (1+t)^{-1+2p_0} \epsilon_0^3. \quad (3.88)$$

Similar to the procedures we did for  $J_{\beta,\gamma}^2(\Phi^\gamma)$ , with minor modifications, we can estimate  $J_\alpha^2$  very similarly. We omit the details here.

#### 4. LINEAR DECAY ESTIMATE

**Lemma 4.1** (Linear Decay Estimate). *For any  $t \in \mathbb{R}$  and any suitable function  $f(x)$ , we have*

$$\|e^{it|\nabla|} f\|_{Z'} \lesssim (1+t)^{-1/2} \|f\|_Z + (1+t)^{-5/8} [\|f\|_{H^{N_0-1}} + \|x|\nabla f\|_{H^{N_1-1}}]. \quad (4.1)$$

*Proof.* To prove (4.1), it would be sufficient to prove the following estimate,

$$\sum_{k \in \mathbb{Z}} 2^{(N_1+4)k_+} \left| \int_{\mathbb{R}^2} e^{it|\xi|+x \cdot \xi} \widehat{f}(\xi) \psi_k(\xi) d\xi \right| \lesssim 1, \quad (4.2)$$

for any  $t \in \mathbb{R}, x \in \mathbb{R}^2$  and function  $f$  that satisfies the following condition,

$$(1+t)^{-1/2} \|(1+|\xi|)^{N_1+6} \widehat{f}(\xi)\|_{L^\infty} + (1+t)^{-5/8} [\|x|\nabla f\|_{H^{N_1-1}} + \|f\|_{H^{N_0-1}}] \sim 1. \quad (4.3)$$

First, we can rule out the very low frequency case by the following estimate,

$$\sum_{2^k \lesssim (1+t)^{-5/4}} 2^{(N_1+4)k_+} \left| \int_{\mathbb{R}^2} e^{it|\xi|+x \cdot \xi} \widehat{f}(\xi) \psi_k(\xi) d\xi \right| \leq \sum_{2^k \lesssim (1+t)^{-5/8}} 2^k \|P_k f\|_{L^2(\mathbb{R}^2)} \lesssim 1.$$

Next, we can use the  $H^{N_0-1}$  norm to rule out the very high frequency case as follows,

$$\begin{aligned} &\sum_{2^k \gtrsim (1+t)^{5/8(N_1-5)}} 2^{(N_1+4)k_+} \left| \int_{\mathbb{R}^2} e^{it|\xi|+x \cdot \xi} \widehat{f}(\xi) \psi_k(\xi) d\xi \right| \\ &\leq \sum_{2^k \gtrsim (1+t)^{5/8(N_1-5)}} 2^{-(N_1-5)k_+} \|P_k f\|_{H^{N_0-1}} \lesssim 1. \end{aligned}$$

From now on, we assume that  $|t| \geq 1$ , otherwise it would be straightforward. When  $|x/t| \leq 0.99$  or  $|x/t| \geq 1.01$ , we do integration by parts with respect to “ $\xi$ ”, which gives us the following estimate,

$$\begin{aligned} 2^{(N_1+4)k_+} \left| \int_{\mathbb{R}^2} e^{it|\xi|+x \cdot \xi} \widehat{f}(\xi) \psi_k(\xi) d\xi \right| &\lesssim |t|^{-1} 2^{-k} 2^{(N_1+4)k_+} \|\widehat{P_k f}(\xi)\|_{L_\xi^1} + |t|^{-1} 2^{(N_1+4)k_+} \|\partial_\xi \widehat{P_k f}(\xi)\|_{L_\xi^1} \\ &\lesssim |t|^{-1} \|P_k f\|_{H^{N_0-1}} + |t|^{-1} 2^{5k_+} \|x|\nabla f\|_{H^{N_1-1}} \lesssim |t|^{-3/8} |t|^{25/8(N_1-5)} \lesssim 1. \end{aligned} \quad (4.4)$$

It remains to consider the case when  $|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}$  and  $0.99 \leq |x/t| \leq 1.01$ . Note that the phase  $\Phi(\xi) = t|\xi| + x \cdot \xi$  has a line of critical points, i.e.,  $\Phi'(\xi) = 0$  if  $\xi/|\xi| = -x/t = -x/|x| =: \xi_0$ . We first localize the angle of  $\xi$  with respect to  $\xi_0$ , and then use the size of support if it is close to the



critical points and do integration by parts in “ $\xi$ ” if it is away from the critical points. Let  $\tilde{l}_k$  be the least integer such that  $2^{\tilde{l}_k} \geq |t|^{-1/2} 2^{-k/2}$ , then

$$\begin{aligned} & \sum_{|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}} 2^{(N_1+4)k_+} \left| \int_{\mathbb{R}^2} e^{it|\xi|+x \cdot \xi} \widehat{f}(\xi) \psi_k(\xi) \psi_{\leq \tilde{l}_k}(\xi/|\xi| - \xi_0) d\xi \right| \\ & \lesssim \sum_{|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}} 2^{2k+\tilde{l}_k} 2^{(N_1+4)k_+} \|\widehat{P_k \psi}\|_{L^\infty} \lesssim \sum_{|t|^{-5/4} \leq 2^k \leq |t|} 2^{3k/2} 2^{-2k_+} \lesssim 1. \end{aligned} \quad (4.5)$$

Note that  $|\nabla \Phi(\xi)| \geq |t| 2^l$  when  $|\xi/|\xi| - \xi_0| \sim 2^l$ , hence after integration by parts in “ $\xi$ ”, we have

$$\begin{aligned} & \sum_{|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}} 2^{(N_1+4)k_+} \sum_{\tilde{l}_k \leq l \leq 2} \left| \int_{\mathbb{R}^2} e^{it|\xi|+x \cdot \xi} \widehat{f}(\xi) \psi_k(\xi) \psi_l(\xi/|\xi| - \xi_0) d\xi \right| \lesssim \\ & \sum_{|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}} 2^{(N_1+4)k_+} \sum_{\tilde{l}_k \leq l \leq 2} \frac{1}{|t| 2^l} (2^{-(k+l)} 2^{2k+l} \|\widehat{P_k f}\|_{L^\infty} + 2^{(2k+l)/2} \|\partial_\xi \widehat{f}(\xi) \psi_k(\xi)\|_{L^2}) \\ & \lesssim \sum_{|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}} \sum_{\tilde{l}_k \leq l \leq 2} \frac{1}{|t|^{1/2}} 2^{-l} 2^{k-2k_+} + \frac{1}{|t|^{3/8}} 2^{-l/2} 2^{5k_+} \\ & \lesssim \sum_{|t|^{-5/4} \leq 2^k \leq |t|^{5/8(N_1-5)}} 2^{3k/2} 2^{-2k_+} + \frac{1}{t^{1/8}} 2^{k/4} 2^{5k_+} \lesssim 1. \end{aligned} \quad (4.6)$$

Combing (4.5) and (4.6), we can see that (4.2) also holds for the remaining cases, therefore finishing the proof.  $\square$

## 5. PROOF OF PROPOSITION 2.4

The proof of Proposition 2.4 is separated into the following two steps:

*Step 1:* Deriving the improved  $Z$ -norm estimates for  $\Phi$ , which further give us the improved dispersion estimate for  $\Phi$ , i.e., prove (2.4).

*Step 2:* Deriving the improved estimate for  $\phi_0$  via the bootstrap argument on the constraint, i.e., prove (2.5).

**5.1. Improved  $Z$ -norm estimate and dispersion estimate for  $\Phi$ .** Recall the equation satisfied by  $\Phi$  in (3.11), we can replace  $\phi_0$  by  $\mathcal{N}_2$  in  $\tilde{Q}_{0,\mu}$  several times until the quartic terms only depend on  $\Phi$ . More precisely, we can reformulate (3.11) as follows,

$$\partial_t \Phi + i|\nabla| \Phi = Q_2 + C + Q_4 + \mathcal{R}, \quad (5.1)$$

where

$$\begin{aligned} Q_2 &= \sum_{(\mu,\nu) \in \mathcal{S}} \tilde{Q}_{\mu,\nu}(\Phi_\mu, \Phi_\nu), \quad \mathcal{R} = \mathcal{N}_1 - Q_2 - C - Q_4, \\ C &= \sum_{\mu,\nu,\kappa \in \{+,-\}} \tilde{Q}_{0,\mu}(\tilde{Q}_{\nu,\kappa}^1(\Phi_\nu, \Phi_\kappa), \Phi_\mu), \end{aligned} \quad (5.2)$$

$$Q_4 = \sum_{\mu,\nu,\kappa,\tau \in \{+,-\}} \tilde{Q}_{0,0}(\tilde{Q}_{\mu,\nu}^1(\Phi_\mu, \Phi_\nu), \tilde{Q}_{\kappa,\tau}^1(\Phi_\nu, \Phi_\kappa)) + \tilde{Q}_{0,\mu}(\tilde{Q}_{0,\nu}^1(\tilde{Q}_{\kappa,\tau}^1(\Phi_\kappa, \Phi_\tau), \Phi_\nu), \Phi_\mu), \quad (5.3)$$

and it is not difficult to see that, in the sense of decay rate, “ $\mathcal{R}$ ” is of quintic and higher.

Define the associated profile of  $\Phi$  as  $g(t) = e^{it|\nabla|}\Phi(t)$ , it follows that

$$\partial_t g(t) = e^{it|\nabla|}[Q_2 + C + Q_4 + \mathcal{R}]. \quad (5.4)$$

Recall the normal form transformation defined in (3.12), we define the associated profile of  $\tilde{\Phi}$  as  $\tilde{g}(t) = e^{it|\nabla|}\tilde{\Phi}$ , it follows that

$$\begin{aligned} \partial_t \tilde{g}(t) = e^{it|\nabla|} & \left[ C + Q_4 + \mathcal{R} + \sum_{(\mu, \nu) \in \mathcal{S}} A_{\mu, \nu}(\Phi_\mu, P_\nu(Q_2 + C + Q_4 + \mathcal{R})) \right. \\ & \left. + A_{\mu, \nu}(P_\mu(Q_2 + C + Q_4 + \mathcal{R}), \Phi_\nu) \right]. \end{aligned} \quad (5.5)$$

From (3.64) and the bootstrap assumption (2.2), we have the following estimates

$$\sup_{t \in [0, T]} (1+t)^{1/2} \|e^{-it|\nabla|}g\|_{Z'} + (1+t)^{-2p_0} \|g\|_Z \lesssim \epsilon_1, \quad \sup_{t \in [0, T]} (1+t)^{-p_0} [\|g\|_{H^{N_0}} + \|\tilde{g}\|_{H^{N_0-1}}] \lesssim \epsilon_0, \quad (5.6)$$

$$\sup_{t \in [0, T]} (1+t)^{-p_0} [\|\mathcal{F}^{-1}[\xi|\nabla_\xi \hat{g}(\xi)](\cdot)\|_{H^{N_1}} + \|\mathcal{F}^{-1}[\xi|\nabla_\xi \hat{\tilde{g}}(\xi)](\cdot)\|_{H^{N_1-1}}] \lesssim \epsilon_0. \quad (5.7)$$

We postpone the proof of estimate (5.7) to the end of this section and take this estimate as granted first.

5.1.1. *Proof of (2.4).* From the results in Lemma 5.1, Lemma 5.2 and Lemma 5.3, we have

$$\begin{aligned} \sup_{t \in [0, T]} (1+t)^{-2p_0} \|g\|_Z & \lesssim \|g(0)\|_Z + \sup_{t \in [0, T]} (1+t)^{-2p_0} \left\| \int_0^t \partial_t g \right\|_Z \lesssim \epsilon_0, \\ \sup_{t \in [0, T]} \|\tilde{g}(t)\|_Z & \lesssim \|g(0)\|_Z + \sup_{t \in [0, T]} \left\| \int_0^t \partial_t g(s) ds + \sum_{(\mu, \nu) \in \mathcal{S}} e^{it|\nabla|} [A_{\mu, \nu}(\Phi_\mu, \Phi_\nu)] \right\|_Z \lesssim \epsilon_0. \end{aligned}$$

From the linear decay estimate (4.1) in Lemma 4.1, we have

$$\sup_{t \in [0, T]} (1+t)^{1/2} \|\tilde{\Phi}(t)\|_{Z'} \lesssim \epsilon_0 + \sup_{t \in [0, T]} (1+t)^{-1/8} [\|\tilde{g}\|_{H^{N_0-1}} + \|\mathcal{F}^{-1}[\xi|\nabla_\xi \hat{\tilde{g}}(\xi)]\|_{H^{N_1-1}}] \lesssim \epsilon_0,$$

which further gives us the following estimate,

$$\begin{aligned} \sup_{t \in [0, T]} (1+t)^{1/2} \|\Phi(t)\|_{Z'} & \lesssim \sup_{t \in [0, T]} (1+t)^{1/2} \|\tilde{\Phi}\|_{Z'} + \sup_{t \in [0, T]} (1+t)^{1/2} \left\| \sum_{(\mu, \nu) \in \mathcal{S}} A_{\mu, \nu}(\Phi_\mu, \Phi_\nu) \right\|_{Z'} \\ & \lesssim \epsilon_0 + \sup_{t \in [0, T]} (1+t)^{1/2} \|\Phi\|_{Z'}^{3/2} \|\Phi\|_{H^{N_0}}^{1/2} \lesssim \epsilon_0. \end{aligned}$$

Therefore (2.4) holds.

5.1.2. *Z-norm estimate for the remainder terms.* We can first estimate the remainder term very easily and have the following lemma:

**Lemma 5.1.** *Under the bootstrap assumption (2.2) and the energy estimate (2.3), we have*

$$\|\mathcal{R}\|_Z \lesssim (1+t)^{-3/2+2p_0} \epsilon_1^5, \quad (5.8)$$

which further gives us

$$\sup_{t \in [0, T]} \left\| \int_0^t e^{is|\nabla|} \mathcal{R} ds \right\|_Z \lesssim \epsilon_0.$$

*Proof.* Since  $\mathcal{R}$  is of quintic and higher, by multilinear estimate, it is easy to derive

$$\begin{aligned} \|\mathcal{R}\|_Z &\lesssim \|\Phi\|_{X_{N_0}}^2 [\|\Phi\|_{Z'}^3 + \|\phi_0\|_{Z'_1} \|\Phi\|_{Z'}] + \|\phi_0\|_{X_{N_0}}^2 \|\Phi\|_{Z'} \\ &+ \|\phi_0\|_{X_{N_0}} \|\Phi\|_{X_{N_0}} (\|\Phi\|_{Z'}^2 + \|\phi_0\|_{Z'_1}) \lesssim (1+t)^{-3/2+2p_0} \epsilon_1^5. \end{aligned}$$

□

**5.1.3.  $Z$ -norm estimate for the cubic and quartic terms.** Next, we proceed to estimate  $C$  and  $Q_4$ , it turns out that we can treat one of the bilinear terms inside  $Q_4$  as a single input and then estimate  $C$  and  $Q_4$  in the same way. More precisely, we have the following lemma,

**Lemma 5.2.** *Under the bootstrap assumption (2.2) and the energy estimate (2.3), we have*

$$\|C\|_Z + \|Q_4\|_Z \lesssim (1+t)^{-7/5-p_0} \epsilon_1^3, \quad (5.9)$$

which gives us the following,

$$\sup_{t \in [0, T]} \left\| \int_0^t e^{it|\nabla|} C ds \right\|_Z + \left\| \int_0^t e^{it|\nabla|} Q_4 ds \right\|_Z \lesssim \epsilon_0^2.$$

*Proof.* it is sufficient to consider the case when  $t \geq 1$ , otherwise, it is trivial. To prove (5.9), essentially speaking, we only need to estimate the following trilinear form in  $Z$ -normed space for any possible signs  $\mu, \nu, \kappa \in \{+, -\}$ ,

$$T(\tilde{Q}_{\mu, \nu}^1(f_\mu, g_\nu), h_\kappa), \quad T \in \{\tilde{Q}_{0, +}(\cdot, \cdot), \tilde{Q}_{0, -}(\cdot, \cdot), \tilde{Q}_{0, 0}(\cdot, \cdot)\}, \quad (5.10)$$

where  $f, g$ , and  $h$  are well defined functions and they satisfy the following estimate,

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|(f, g, h)\|_{X_{N_0}} + (1+t)^{1/2} \|(f, g, h)\|_{Z'} + (1+t)^{-2p_0} \|(f, g, h)\|_Z \lesssim \epsilon_1. \quad (5.11)$$

Define the associated profiles of  $f, g$ , and  $h$  as  $\tilde{f}(t) := e^{it|\nabla|} f(t)$ ,  $\tilde{g}(t) := e^{it|\nabla|} g(t)$ , and  $\tilde{h}(t) := e^{it|\nabla|} h(t)$  respectively. Write above trilinear form on the Fourier side, we have

$$\mathcal{F}[T(\tilde{Q}_{\mu, \nu}^1(f_\mu, g_\nu), h_\kappa)](\xi) = \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\Phi_{\mu, \nu, \kappa}(\xi, \eta, \sigma)} \widehat{\tilde{f}}_\mu(\xi - \sigma) \widehat{\tilde{g}}_\nu(\sigma - \eta) \widehat{\tilde{h}}_\kappa(\eta) m_{\mu, \nu}(\xi, \eta, \sigma) d\eta d\sigma,$$

where

$$\Phi_{\mu, \nu, \kappa}(\xi, \eta, \sigma) = -\mu|\xi - \sigma| - \nu|\sigma - \eta| - \kappa|\eta|, \quad m_{\mu, \nu}(\xi, \eta, \sigma) = \tilde{m}_{\mu, \nu}^1(\xi - \sigma, \sigma - \eta) t(\xi - \eta, \eta),$$

and  $t(\cdot, \cdot)$  is the associated symbol of the bilinear operator  $T(\cdot, \cdot)$ .

Now we are ready to do  $Z$ -norm estimate and we can first rule out the very high frequency case by multilinear estimate as follows,

$$\begin{aligned} &\sup_{2^k \geq (1+t)^{1/N_1}} \|P_k [T(\tilde{Q}_{\mu, \nu}^1(f_\mu, g_\nu), h_\kappa)]\|_Z \\ &\lesssim \sup_{2^k \geq (1+t)^{1/N_1}} 2^{-(N_0 - N_1 - 10)k} \|(f, g, h)\|_{H_{N_0}}^2 \|(f, g, h)\|_{Z'} \lesssim (1+t)^{-7/5-10p_0} \epsilon_1^3. \end{aligned} \quad (5.12)$$

For the remaining case, we do integration by parts in “ $\sigma$ ”. More precisely, after integrating by parts in  $\sigma$ , we have

$$\left| \mathcal{F}[T(\tilde{Q}_{\mu, \nu}^1(P_\mu f, P_\nu g), P_\kappa h)](\xi) \right| \lesssim \frac{1}{t} [|I_{\mu, \nu, \kappa}^1(\xi)| + |I_{\mu, \nu, \kappa}^2(\xi)| + |I_{\mu, \nu, \kappa}^3(\xi)|], \quad (5.13)$$

where

$$\begin{aligned}
I_{\mu,\nu,\kappa}^1(\xi) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\Phi_{\mu,\nu,\kappa}(\xi,\eta,\sigma)} \widehat{f}_\mu(\xi - \sigma) \widehat{g}_\nu(\sigma - \eta) \widehat{h}_\kappa(\eta) \nabla_\sigma \cdot \widehat{m}_{\mu,\nu}(\xi, \eta, \sigma) d\eta d\sigma, \\
I_{\mu,\nu,\kappa}^2(\xi) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\Phi_{\mu,\nu,\kappa}(\xi,\eta,\sigma)} \widehat{f}_\mu(\xi - \sigma) \nabla_\sigma \cdot \widehat{g}_\nu(\sigma - \eta) \widehat{m}_{\mu,\nu}(\xi, \eta, \sigma) \widehat{h}_\kappa(\eta) d\eta d\sigma, \\
I_{\mu,\nu,\kappa}^3(\xi) &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{it\Phi_{\mu,\nu,\kappa}(\xi,\eta,\sigma)} \nabla_\sigma \widehat{f}_\mu(\xi - \sigma) \cdot \widehat{m}_{\mu,\nu}(\xi, \eta, \sigma) \widehat{g}_\nu(\sigma - \eta) \widehat{h}_\kappa(\eta) d\eta d\sigma, \\
\widehat{m}_{\mu,\nu}(\xi, \eta, \sigma) &= \frac{\nabla_\sigma \Phi_{\mu,\nu,\kappa}(\xi, \eta, \sigma)}{|\nabla_\sigma \Phi_{\mu,\nu,\kappa}(\xi, \eta, \sigma)|^2} m_{\mu,\nu}(\xi, \eta, \sigma) = \frac{\mu \frac{\xi - \sigma}{|\xi - \sigma|} - \nu \frac{\sigma - \eta}{|\sigma - \eta|}}{\left| \mu \frac{\xi - \sigma}{|\xi - \sigma|} - \nu \frac{\sigma - \eta}{|\sigma - \eta|} \right|^2} \tilde{m}_{\mu,\nu}^1(\xi - \sigma, \sigma - \eta) t(\xi - \eta, \eta).
\end{aligned}$$

Recall that there is a strong null structure inside the symbol  $\tilde{m}_{\mu,\nu}^1(\cdot, \cdot)$  (see (3.7) and (3.8)), in any case, we can gain one degree of angle between  $\xi - \sigma$  and  $\sigma - \eta$ , which compensates the loss in denominator.

From Lemma 2.1, Lemma 2.2 and the discussion in subsection 3.3, we can verify that the following estimates hold,

$$\|\mathcal{F}^{-1}[\widehat{m}_{\mu,\nu}(\xi, \eta, \sigma) \psi_k(\xi) \psi_{k_1}(\xi - \eta) \psi_{k_2}(\eta) \psi_{k'_1}(\xi - \sigma) \psi_{k'_2}(\sigma - \eta)]\|_{L^1} \lesssim 2^{\min\{k'_1, k'_2\} + k_1 + k_2}, \quad (5.14)$$

$$\|\mathcal{F}^{-1}[\nabla_\sigma \cdot \widehat{m}_{\mu,\nu}(\xi, \eta, \sigma) \psi_k(\xi) \psi_{k_1}(\xi - \eta) \psi_{k_2}(\eta) \psi_{k'_1}(\xi - \sigma) \psi_{k'_2}(\sigma - \eta)]\|_{L^1} \lesssim 2^{\max\{k'_1, k'_2\} + k_2}. \quad (5.15)$$

Hence, we have

$$\begin{aligned}
\sup_{2^k \leq (1+t)^{1/N_1}} \sum_{i=1,2,3} \|\mathcal{F}^{-1}[I_{\mu,\nu,\kappa}^i(\cdot) \psi_k(\cdot)]\|_Z &\lesssim (1+t)^{10/N_1} [\|(f, g, h)\|_{H^{N_0}}^2 + \|\xi| \nabla_\xi(\widehat{f}(\xi), \widehat{g}(\xi))\|_{H^{N_1}}] \times \\
\|(f, g, h)\|_{Z'} &\lesssim (1+t)^{10/N_1} \|(f, g, h)\|_{X_{N_0}}^2 \|(f, g, h)\|_{Z'} \lesssim (1+t)^{-2/5-10p_0} \epsilon_1^3. \quad (5.16)
\end{aligned}$$

To sum up, after combining (5.12), (5.13) and (5.16), we can see the following estimate holds under the smallness assumption (5.11),

$$\|T(\tilde{Q}_{\mu,\nu}^1(P_\mu f, P_\nu g), P_\kappa h)\|_Z \lesssim (1+t)^{-7/5-10p_0} \epsilon_1^3. \quad (5.17)$$

From the explicit formula of “ $C$ ” in (5.2) and the bootstrap assumption, we can immediately derive the improved  $Z$ -norm estimate for “ $C$ ”.

Let us proceed to consider the quintic term  $Q_4$ . Recall the explicit formula of  $Q_4$  in (5.3) and then let  $h := \tilde{Q}_{\kappa,\tau}^1(\Phi_\nu, \Phi_\kappa)$ , we have

$$\tilde{Q}_{0,0}(\tilde{Q}_{\mu,\nu}^1(\Phi_\mu, \Phi_\nu), \tilde{Q}_{\kappa,\tau}^1(\Phi_\nu, \Phi_\kappa)) = \tilde{Q}_{0,0}(\tilde{Q}_{\mu,\nu}^1(\Phi_\mu, \Phi_\nu), h),$$

and

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|h\|_{X_{N_0}} + (1+t)^{1/2} \|h\|_{Z'} + (1+t)^{-2p_0} \|h\|_Z \lesssim \epsilon_1.$$

Therefore, we can use the derived general type  $Z$ -norm estimate (5.17) for the first term on the right hand side of (5.3). For the second term on the right hand side of (5.3), we can treat the trilinear form of type (5.10) as a input of bilinear operator  $\tilde{Q}_{0,\mu}(\cdot, \cdot)$ , then the  $Z$ -norm estimate will be straightforward. To sum up, we have

$$\|Q_4\|_Z \lesssim (1+t)^{-7/5-10p_0} \epsilon_1^3 + (1+t)^{-7/5-10p_0} \epsilon_1^3 \|\Phi\|_{H_{N_0}} \lesssim (1+t)^{-7/5-p_0} \epsilon_1^3,$$

which infers that (5.9) holds, hence finishing the proof.  $\square$

5.1.4. *Z-norm estimate for the quadratic terms.* Lastly, we consider the quadratic terms and we have the following lemma.

**Lemma 5.3.** *Under the bootstrap assumption (2.2) and the energy estimate (2.3), we have*

$$\sup_{t \in [0, T]} (1+t)^{-2p_0} \left\| \int_0^t e^{is|\nabla|} Q_2 ds \right\|_Z \lesssim \epsilon_0. \quad (5.18)$$

$$\sup_{t \in [0, T]} \left\| \int_0^t e^{is|\nabla|} Q_2 ds + \sum_{(\mu, \nu) \in \mathcal{S}} e^{it\Lambda} [A_{\mu, \nu}(\Phi_\mu, \Phi_\nu)] \right\|_Z \lesssim \epsilon_0. \quad (5.19)$$

*Proof.* We write  $Q_2$  on the Fourier side in terms of profile  $g$  and have the following,

$$\mathcal{F}\left(\int_0^t e^{is|\nabla|} Q_2 ds\right)(\xi) = \sum_{(\mu, \nu) \in \mathcal{S}} \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu, \nu}(\xi, \eta)} \widehat{g}_\mu(s, \xi - \eta) \widehat{g}_\nu(s, \eta) \tilde{m}'_{\mu, \nu}(\xi - \eta, \eta) d\eta ds. \quad (5.20)$$

For this case, we do integration by parts in time and have the following identity,

$$\mathcal{F}\left(\int_0^t e^{is|\nabla|} Q_2 ds\right)(\xi) = \sum_{(\mu, \nu) \in \mathcal{S}} \mathcal{J}_1^{\mu, \nu}(\xi) + \mathcal{J}_2^{\mu, \nu}(\xi) + \mathfrak{E}nd_1^{\mu, \nu}(\xi) - \mathfrak{E}nd_0^{\mu, \nu}(\xi), \quad (5.21)$$

where

$$\begin{aligned} \mathcal{J}_1^{\mu, \nu}(\xi) &= \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu, \nu}(\xi, \eta)} \partial_s \widehat{g}_\mu(s, \xi - \eta) \widehat{g}_\nu(s, \eta) a_{\mu, \nu}(\xi - \eta, \eta) d\eta ds, \\ \mathcal{J}_2^{\mu, \nu}(\xi) &= \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu, \nu}(\xi, \eta)} \widehat{g}_\mu(s, \xi - \eta) \partial_s \widehat{g}_\nu(s, \eta) a_{\mu, \nu}(\xi - \eta, \eta) d\eta ds, \\ \mathfrak{E}nd_1^{\mu, \nu}(\xi) &= - \int_{\mathbb{R}^2} e^{it\Phi_{\mu, \nu}(\xi, \eta)} \widehat{g}_\mu(t, \xi - \eta) \widehat{g}_\nu(t, \eta) a_{\mu, \nu}(\xi - \eta, \eta) d\eta = - \mathcal{F}[e^{it|\nabla|} A_{\mu, \nu}(\Phi_\mu(t), \Phi_\nu(t))](\xi), \\ \mathfrak{E}nd_0^{\mu, \nu}(\xi) &= - \mathcal{F}[A_{\mu, \nu}(\Phi_\mu(0), \Phi_\nu(0))](\xi). \end{aligned} \quad (5.22)$$

Combing (5.22) and (5.21), we also have the following equality,

$$\begin{aligned} &\mathcal{F}\left(\int_0^t e^{is|\nabla|} Q_2 ds\right)(\xi) + \sum_{(\mu, \nu) \in \mathcal{S}} \mathcal{F}\left[e^{it\Lambda} A_{\mu, \nu}(\Phi_\mu(t), \Phi_\nu(t))\right](\xi) \\ &= \sum_{(\mu, \nu) \in \mathcal{S}} \mathcal{F}\left[A_{\mu, \nu}(\Phi_\mu(0), \Phi_\nu(0))\right](\xi) + \mathcal{J}_1^{\mu, \nu}(\xi) + \mathcal{J}_2^{\mu, \nu}(\xi). \end{aligned} \quad (5.23)$$

From  $L^2 - L^2$  type estimate, we can estimate the endpoint cases as follows,

$$\|\mathcal{F}^{-1}(\mathfrak{E}nd_1^{\mu, \nu}(\cdot))\|_Z \lesssim \|g(t)\|_{H^{N_0}}^2 \lesssim (1+t)^{2p_0} \epsilon_1^2 \lesssim (1+t)^{2p_0} \epsilon_0, \quad \|\mathcal{F}^{-1}(\mathfrak{E}nd_0^{\mu, \nu}(\cdot))\|_Z \lesssim \epsilon_0. \quad (5.24)$$

Let's proceed to estimate  $\mathcal{J}_1^{\mu, \nu}(\xi)$ . We can plug in the equation satisfied by  $\partial_t g$  (see (5.4)) and have the following

$$\mathcal{J}_1^{\mu, \nu}(\xi) = \mathcal{C}^1 + \mathcal{Q}_4^1 + \mathcal{R}^1,$$

where

$$\mathcal{Q}_4^1 = \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu, \nu}(\xi, \eta)} P_\mu[\widehat{e^{is|\nabla|} C}](s, \xi - \eta) \widehat{g}_\nu(s, \eta) a_{\mu, \nu}(\xi - \eta, \eta) d\eta ds,$$

$$\begin{aligned}\mathcal{R}^1 &= \int_0^t \int_{\mathbb{R}^2} e^{is\Phi_{\mu,\nu}(\xi,\eta)} P_\mu[\widehat{e^{is|\nabla|}[Q_4 + \mathcal{R}]}](s, \xi - \eta) \widehat{g}_\nu(s, \eta) a_{\mu,\nu}(\xi - \eta, \eta) d\eta ds, \\ \mathcal{C}^1 &= \sum_{k_1, k_2 \in \mathbb{Z}} \sum_{(\kappa, \tau) \in \mathcal{S}} \int_0^t \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} e^{is\Phi_{\mu,\nu}^{\kappa,\tau}(\xi, \eta, \sigma)} \widehat{P_\mu[g_\kappa]}(s, \xi - \sigma) \times \\ &\quad \widehat{P_\mu[g_\tau]}(s, \sigma - \eta) \widehat{g}_\nu(s, \eta) b_{\mu,\nu,\kappa,\tau}^{k_1, k_2}(\xi, \eta, \sigma) d\eta d\sigma ds,\end{aligned}$$

and

$$\Phi_{\mu,\nu}^{\kappa,\tau}(\xi, \eta, \sigma) = |\xi| - \mu\kappa|\xi - \sigma| - \mu\tau|\sigma - \eta| - \nu|\eta|,$$

$$b_{\mu,\nu,\kappa,\tau}^{k_1, k_2}(\xi, \eta, \sigma) = \tilde{m}'_{\kappa,\tau}(\xi - \sigma, \sigma - \eta) a_{\mu,\nu}(\xi - \eta, \eta) \psi_{k_1}(\xi - \eta) \psi_{k_2}(\eta).$$

As  $\mathcal{R}^1$  is of quintic and higher, we can estimate it in the same way as we did for  $\mathcal{R}$  in Lemma 5.8. Moreover, we can estimate  $\mathcal{Q}_4^1$  in the same way as we did for  $C$  and  $Q_4$  in Lemma 5.2, because of the presence of bilinear operator  $\widehat{Q}_{\mu,\nu}^1(\cdot, \cdot)$  in the term  $\mathcal{Q}_4^1$ . We omit the details for those cases here. To sum up, we have

$$\|\mathcal{F}^{-1}[\mathcal{Q}_4^1]\|_Z + \|\mathcal{F}^{-1}[\mathcal{R}^1]\|_Z \lesssim 2^{-p_0 m} \epsilon_1^3 \lesssim 2^{-p_0 m} \epsilon_0^2.$$

It remains to estimate  $\mathcal{C}^1$ . We can use *integration by parts in “ $\sigma$ ”*, which is used to estimate “ $C$ ” in the proof of Lemma 5.2, to estimate all  $\mathcal{C}^1$  without any problem. To see this point, we first define

$$\widehat{b}_{\mu,\nu,\kappa,\tau}^{k_1, k_2}(\xi, \eta, \sigma) = \frac{\nabla_\sigma \Phi_{\mu,\nu}^{\kappa,\tau}(\xi, \eta, \sigma)}{|\nabla_\sigma \Phi_{\mu,\nu}^{\kappa,\tau}(\xi, \eta, \sigma)|^2} b_{\mu,\nu,\kappa,\tau}^{k_1, k_2}(\xi, \eta, \sigma),$$

and then we claim the following two estimates hold, which are sufficient to close the argument,

$$\|\mathcal{F}^{-1}[\widehat{b}_{\mu,\nu,\kappa,\tau}^{k_1, k_2}(\xi, \eta, \sigma) \psi_k(\xi) \psi_{k'}(\xi - \sigma) \psi_{k'_2}(\sigma - \eta)]\|_{L^1} \lesssim 2^{k'_1 + k'_2 + \max\{k_1, k_2\}}, \quad (5.25)$$

$$\|\mathcal{F}^{-1}[\nabla_\sigma \cdot \widehat{b}_{\mu,\nu,\kappa,\tau}^{k_1, k_2}(\xi, \eta, \sigma) \psi_k(\xi) \psi_{k'}(\xi - \sigma) \psi_{k'_2}(\sigma - \eta)]\|_{L^1} \lesssim 2^{\max\{k'_1, k'_2\} + \max\{k_1, k_2\}}. \quad (5.26)$$

To prove (5.25) and (5.26), we can use the Lemma 2.2 and the discussion in subsection 3.3 safely for the Low  $\times$  High interaction and High  $\times$  Low interaction. It remains to check for the High  $\times$  High interaction case, more precisely,  $|\xi - \eta| \ll |\xi - \sigma| \sim |\sigma - \eta|$ .

For this case, note that if  $\kappa$  and  $\tau$  have the same sign then  $\sigma - \xi$  and  $\sigma - \xi + \xi - \eta = \sigma - \eta$  are almost in the same direction, hence

$$|\nabla_\sigma \Phi_{\mu,\nu}^{\kappa,\tau}(\xi, \eta, \sigma)| = \left| \frac{\sigma - \xi}{|\sigma - \xi|} + \frac{\sigma - \eta}{|\sigma - \eta|} \right| \sim 1,$$

and estimates (5.25) and (5.7) hold. It remains to check the case when  $\kappa$  and  $\tau$  have different sign, i.e.,  $(\kappa, \tau) = (+, -)$  and we have

$$|\nabla_\sigma \Phi_{\mu,\nu}^{\kappa,\tau}(\xi, \eta, \sigma)| = \left| \frac{\sigma - \xi}{|\sigma - \xi|} - \frac{\sigma - \eta}{|\sigma - \eta|} \right|.$$

From (3.10), it is easy to see that, the  $(1 + \cos(\xi - \sigma, \sigma - \eta))$  part of  $\tilde{m}'_{+,-}(\xi - \sigma, \sigma - \eta)$  is sufficient to compensate the denominator part. Hence the size of symbol is not big even if  $\xi - \eta$  is very small. We can use the Lemma 2.2 to show that estimates (5.25) and (5.26) still hold for the High  $\times$  High interaction.  $\square$

### 5.2. Improved estimate for $\phi_0$ via bootstrap argument on the constraint.

**Lemma 5.4.** *With the improved estimate we have proven for  $\Phi$  as follows,*

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|(\phi_0, \Phi)\|_{X_{N_0}} + (1+t)^{1/2} \|\Phi\|_{Z'} \lesssim \epsilon_0, \quad (5.27)$$

*we have the following improved estimate for  $\phi_0$ ,*

$$\sup_{t \in [0, T]} (1+t)^{1/2-p_0} \|\phi_0\|_{X_{N_0}} + (1+t) \|\phi_0\|_{Z'_1} \lesssim (\epsilon_0 + \epsilon_1)^2 \lesssim \epsilon_0^2. \quad (5.28)$$

*Proof.* From the constraint equation  $\phi_0 = \mathcal{N}_2$  and the estimates in Lemma 3.4, we have the following estimates for fixed  $t \in [0, T]$ ,

$$\begin{aligned} \|\phi_0(t)\|_{X_{N_0}} &\lesssim (\|\phi_0\|_{X_{N_0}} + \|\Phi\|_{X_{N_0}}) (\|\phi_0\|_{Z'_1} + \|\Phi\|_{Z'}) \lesssim \frac{1}{(1+t)^{1/2-p_0}} (\epsilon_0 + \epsilon_1)^2, \\ \|\phi_0(t)\|_{Z'_1} &\lesssim \|\Phi\|_{Z'}^2 + \|\phi_0\|_{Z'_1} \|\Phi\|_{Z'} + (\|\phi_0\|_{Z'_1} \|\Phi\|_{Z'})^{3/4} \|(\phi_0, \Phi)\|_{X_{N_0}}^{1/4} + \|\phi_0\|_{Z'_1}^2 + \|\phi_0\|_{Z'_1}^{3/2} \|\phi_0\|_{X_{N_0}}^{1/2} \\ &\lesssim \left( \frac{1}{1+t} + \frac{1}{(1+t)^{9/8-p_0/4}} \right) (\epsilon_0 + \epsilon_1)^2 \lesssim \frac{1}{1+t} (\epsilon_0 + \epsilon_1)^2. \end{aligned}$$

Therefore (5.28) holds.  $\square$

### 5.3. Proof of (5.7).

We assume that  $|t| \geq 1$ , otherwise it's trivial. Note that

$$\begin{aligned} \|\mathcal{F}^{-1}[\xi|\nabla_\xi \widehat{g}(\xi)](\cdot)\|_{H^{N_1}} &\lesssim \|Sg\|_{H^{N_1}} + \|\Omega g\|_{H^{N_1}} + \|t\partial_t g\|_{H^{N_1}}, \\ Sg &= e^{it|\nabla|} S\Phi, \quad \Omega g = e^{it|\nabla|} \Omega\Phi, \end{aligned} \quad (5.29)$$

and we have very similar estimates for  $\tilde{g}$ . Hence it's sufficient to estimate  $t\partial_t g$  in  $H^{N_1}$  and  $t\partial_t \tilde{g}$  in  $H^{N_1-1}$ . Recall the equation satisfied by  $\tilde{g}$  in (5.5), due to the cubic and higher structure, it is not difficult to derive the following estimate,

$$\sup_{t \in [0, T]} (1+t)^{-p_0} \|\mathcal{F}^{-1}[\xi|\nabla_\xi \widehat{g}(\xi)]\|_{H^{N_1-1}} \lesssim \epsilon_0.$$

Recall the equation satisfied by  $g$  in (5.4), we have

$$\|e^{it|\nabla|}[C + Q_4 + R]\|_{H^{N_1}} \lesssim \|(\phi_0, \Phi)\|_{X_{N_0}} (\|\Phi\|_{Z'}^2 + \|\phi_0\|_{Z'_1}) \lesssim (1+t)^{-1+p_0} \epsilon_1^3.$$

It remains to estimate  $Q_2$  in  $H^{N_1}$  norm. We can first rule out the very high frequency as follows,

$$\sum_{2^k \lesssim (1+t)^{2/N_1}} \|P_k[Q_2]\|_{H^{N_1}} \lesssim (1+t)^{-1} \|\Phi\|_{H^{N_0}} \|\Phi\|_{Z'} \lesssim \frac{1}{(1+t)^{3/2}} \epsilon_1^2.$$

For the remaining cases, we do integration by parts in “ $\eta$ ”. Very similar to the proof of estimate (5.25), we have the following estimate,

$$\left\| \frac{\mu \frac{\xi-\eta}{|\xi-\eta|} - \nu \frac{\eta}{|\eta|}}{\left| \mu \frac{\xi-\eta}{|\xi-\eta|} - \nu \frac{\eta}{|\eta|} \right|^2} \tilde{m}'_{\mu, \nu}(\xi - \eta, \eta) \right\|_{S_{k, k_1, k_2}^\infty} \lesssim 2^{k_1+k_2},$$

which further gives us the following estimate

$$\sum_{2^k \lesssim (1+t)^{2/N_1}} \|P_k[Q_2]\|_{H^{N_1}} \lesssim \frac{1}{t} (1+t)^{2/N_1} \|\mathcal{F}^{-1}[\xi|\nabla_\xi \widehat{g}(\xi)]\|_{H^{N_1}} \|e^{-it|\nabla|} g\|_{Z'}$$

$$\lesssim \frac{1}{(1+t)^{5/4}} \epsilon_1 \|\mathcal{F}^{-1}[\xi |\nabla_\xi \widehat{g}(\xi)]\|_{H^{N_1}}.$$

To sum up, we have

$$\|\mathcal{F}^{-1}[\xi |\nabla_\xi \widehat{g}(\xi)](\cdot)\|_{H^{N_1}} \lesssim t^{p_0} \epsilon_0 + \epsilon_1 \|\mathcal{F}^{-1}[\xi |\nabla_\xi \widehat{g}(\xi)]\|_{H^{N_1}},$$

which further gives us

$$\|\mathcal{F}^{-1}[\xi |\nabla_\xi \widehat{g}(\xi)](\cdot)\|_{H^{N_1}} \lesssim t^{p_0} \epsilon_0.$$

Now, we can see the estimate (5.7) indeed holds.

## 6. ASYMPTOTIC BEHAVIOR OF THE SOLUTION

As a byproduct of the global existence result, we can very easily see that  $\phi_0$  scatters to zero in  $X_{N_0}$ . From (5.4), (5.23) and the definition of  $\tilde{g}(t)$ , we are motivated to define

$$\begin{aligned} \tilde{g}_\infty = \tilde{g}(0) + \mathcal{F}^{-1} \Big[ & \int_0^\infty \int_{\mathbb{R}^2} e^{is\Phi_{\mu,\nu}(\xi,\eta)} \partial_s [\widehat{g}_\mu(s, \xi - \eta) \widehat{g}_\nu(s, \eta)] a_{\mu,\nu}(\xi - \eta, \eta) d\eta ds \Big] \\ & + \int_0^\infty e^{is\Lambda} [C + Q_4 + \mathcal{R}] ds, \end{aligned}$$

then as a byproduct of the improved  $Z$ -norm estimate for  $g$ , we have

$$\|\tilde{g}(t) - \tilde{g}_\infty\|_Z \lesssim \frac{1}{(1+t)^{p_0}} \epsilon_0. \quad (6.1)$$

Thus from (6.1), we can easily derive the following,

$$\begin{aligned} \|\Phi(t) - e^{-it|\nabla|} \tilde{g}_\infty\|_{H^{N_1+4}} & \lesssim \|\tilde{\Phi}(t) - e^{-it|\nabla|} \tilde{g}_\infty\|_{H^{N_1+4}} + \sum_{(\mu,\nu) \in \mathcal{S}} \|A_{\mu,\nu}(\Phi_\mu, \Phi_\nu)\|_{H^{N_1+4}} \\ & \lesssim \frac{1}{(1+t)^{p_0}} \epsilon_0 \rightarrow 0, \quad \text{as } |t| \rightarrow \infty. \end{aligned} \quad (6.2)$$

That is to say,  $\Phi(t)$  scatters to a linear solution in a lower regularity Sobolev space.

## APPENDIX: DERIVATION OF SYSTEM (1.9) FROM SYSTEM (1.7)

Recall that

$$v = (-\partial_2 \psi, \partial_1 \psi), \quad G_{\cdot,1} = (-\partial_2 G_1, \partial_1 G_1), \quad G_{\cdot,2} = (-\partial_2 G_2, \partial_1 G_2),$$

it is easy to see the following identities hold,

$$\begin{aligned} \nabla v &= \begin{pmatrix} -\partial_1 \partial_2 \psi & -\partial_2^2 \psi \\ \partial_1^2 \psi & \partial_2 \partial_1 \psi \end{pmatrix}, \quad G = \begin{pmatrix} -\partial_2 G_1 & -\partial_2 G_2 \\ \partial_1 G_1 & \partial_1 G_2 \end{pmatrix}, \\ G^\top &= \begin{pmatrix} -\partial_2 G_1 & \partial_1 G_1 \\ -\partial_2 G_2 & \partial_1 G_2 \end{pmatrix}, \quad GG^\top = \begin{pmatrix} (\partial_2 G_1)^2 + (\partial_2 G_2)^2 & -\partial_2 G_1 \partial_1 G_1 - \partial_2 G_2 \partial_1 G_2 \\ -\partial_1 G_1 \partial_2 G_1 - \partial_1 G_2 \partial_2 G_2 & (\partial_1 G_1)^2 + (\partial_1 G_2)^2 \end{pmatrix}. \end{aligned}$$

We can take the first component of the first equation of the system (1.7) and write it in terms of  $\psi$ ,  $G_1$ , and  $G_2$ . As a result, we have

$$\begin{aligned} -\partial_t \partial_2 \psi + \partial_1 \partial_2 G_1 + \partial_2^2 G_2 &= -\partial_1 p - (-\partial_2 \psi \partial_1 + \partial_1 \psi \partial_2)(-\partial_2 \psi) \\ &\quad + \partial_1 [(\partial_2 G_1)^2 + (\partial_2 G_2)^2] + \partial_2 [-\partial_2 G_1 \partial_1 G_1 - \partial_2 G_2 \partial_1 G_2] \end{aligned}$$



$$\begin{aligned}
&= -\partial_1 p - (\partial_2 \psi \partial_1 \partial_2 \psi - \partial_1 \psi \partial_2^2 \psi) + \partial_2 G_1 \partial_1 \partial_2 G_1 - \partial_1 G_1 \partial_2^2 G_1 + \partial_2 G_2 \partial_1 \partial_2 G_2 - \partial_2^2 G_2 \partial_1 G_2 \\
&= -\partial_1 p - Q_{1,2}(\partial_2 \psi, \psi) + Q_{1,2}(\partial_2 G_1, G_1) + Q_{1,2}(\partial_2 G_2, G_2).
\end{aligned} \tag{6.3}$$

Very similarly, after taking the second component of the first equation of the system (1.7), we have

$$\begin{aligned}
&\partial_t \partial_1 \psi - [\partial_1 \partial_1 G_1 + \partial_2 \partial_1 G_2] = -\partial_2 p - (-\partial_2 \psi \partial_1 + \partial_1 \psi \partial_2)(\partial_1 \psi) \\
&\quad + \partial_1 [-\partial_1 G_1 \partial_2 G_1 - \partial_1 G_2 \partial_2 G_2] + \partial_2 [(\partial_1 G_1)^2 + (\partial_1 G_2)^2] \\
&= -\partial_2 p + Q_{1,2}(\partial_1 \psi, \psi) - Q_{1,2}(\partial_1 G_1, G_1) - Q_{1,2}(\partial_1 G_2, G_2).
\end{aligned} \tag{6.4}$$

Applying  $\partial_2/|\nabla|^2$  on both hands side of equation (6.3) and  $-\partial_1/|\nabla|^2$  on both hands side of equation (6.4), and then adding those two equations together, we have

$$\begin{aligned}
&\partial_t \psi - \partial_1 G_1 - \partial_2 G_2 = -|\nabla|^{-1} R_2 [Q_{1,2}(\partial_2 \psi, \psi) - Q_{1,2}(\partial_2 G_1, G_1) - Q_{1,2}(\partial_2 G_2, G_2)] \\
&\quad - |\nabla|^{-1} R_1 [Q_{1,2}(\partial_1 \psi, \psi) - Q_{1,2}(\partial_1 G_1, G_1) - Q_{1,2}(\partial_1 G_2, G_2)].
\end{aligned}$$

Therefore the first equation of the system (1.9) holds. Now we proceed to derive the equations satisfied by  $G_1$  and  $G_2$ . From the equation satisfied by  $G$  in (1.7), we have the following four equations,

$$\begin{aligned}
&-\partial_t \partial_2 G_1 + \partial_1 \partial_2 \psi = -(-\partial_2 \psi \partial_1 + \partial_1 \psi \partial_2)(-\partial_2 G_1) + (\partial_1 \partial_2 \psi \partial_2 G_1 - \partial_2^2 \psi \partial_1 G_1) \\
&= Q_{1,2}(\psi, \partial_2 G_1) + Q_{1,2}(\partial_2 \psi, G_1) = \partial_2 [Q_{1,2}(\psi, G_1)],
\end{aligned} \tag{6.5}$$

$$\partial_t \partial_1 G_1 - \partial_1^2 \psi = Q_{1,2}(\partial_1 G_1, \psi) + Q_{1,2}(G_1, \partial_1 \psi) = -\partial_1 [Q_{1,2}(\psi, G_1)], \tag{6.6}$$

$$\begin{aligned}
&-\partial_t \partial_2 G_2 + \partial_2^2 \psi = -(-\partial_2 \psi \partial_1 + \partial_1 \psi \partial_2)(-\partial_2 G_2) + (\partial_1 \partial_2 \psi \partial_2 G_2 - \partial_2^2 \psi \partial_1 G_2) \\
&= Q_{1,2}(\psi, \partial_2 G_2) + Q_{1,2}(\partial_2 \psi, G_2) = \partial_2 [Q_{1,2}(\psi, G_2)],
\end{aligned} \tag{6.7}$$

$$\partial_t \partial_1 G_2 - \partial_2 \partial_1 \psi = Q_{1,2}(\partial_1 G_2, \psi) + Q_{1,2}(G_2, \partial_1 \psi) = -\partial_1 [Q_{1,2}(\psi, G_2)]. \tag{6.8}$$

From (6.5) and (6.6), we can see the following equation holds,

$$\partial_t G_1 - \partial_1 \psi = -Q_{1,2}(\psi, G_1) = Q_{1,2}(G_1, \psi).$$

From (6.7) and (6.8), we can see the following equation holds,

$$\partial_t G_2 - \partial_2 \psi = -Q_{1,2}(\psi, G_2) = Q_{1,2}(G_2, \psi).$$

To sum up, all equations in the system (1.9) hold.

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